The Laplace Transform: Convolution Theorem

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Abstract

The Laplace transform is the modern darling of the mathematical methods used by today's engineers. However, to greatly extend the usefulness of this method, we find the beautiful *Convolution Theorem*, which appears to me as though some entity had predetermined that it should fit neatly into the subject of the Laplace transform designed to widen its usefulness.

1 Introduction

The mathematics of the convolution theorem is not too advanced. All we need is some proficiency at multiple integrals and change of ordering of the variables of integration.

So, what is the Laplace transform? In engineering practice, one thinks of it as a means to transfer from the time domain of variable to the frequency domain. But for our purposes here, we will not bother to strongly type the variables we use. Anyway, given a function f defined smoothly on the nonnegative horizontal axis t, we define the Laplace transform, of $f(t)$ by

$$\mathscr{L}[\, f(t)\,] = \int_0^\infty e^{-st} f(t)\, dt \equiv F(s)\,, \tag{1}$$

where $\mathscr{L}$ represent the Laplace operator.

Note that on the RHS of (1) the variable t will be 'integrated out', meaning that the result will not be a function of t, but rather of s. So, let's rewrite (1) as

$$\mathscr{L}_s[\, f(t)\,] = \int_0^\infty e^{-sv} f(v)\, dv\,, \tag{2}$$

where all we did was to substitute v for t. In other words, the variable of integration on the RHS is dummy, that is, is replaceable by some other variable symbol, so long as that new variable is independent of s.

We define the *convolution* of two functions $f(t)$ and $g(t)$ by

$$f(t) \star g(t) \equiv \int_0^t f(u) g(t-u)\, du\,. \tag{3}$$

The LHS can also be represented as

$$f(t) \star g(t) = (f \star g)(t)\,. \tag{4}$$

2 The Convolution Theorem

Let

$$\begin{aligned} F(s) &\equiv \mathscr{L}_s[\, f(t)\,]\,, \\ G(s) &\equiv \mathscr{L}_s[\, G(t)\,]\,. \end{aligned} \tag{5}$$

Normally, the s on the $\mathscr{L}$ is suppressed. Then, the theorem is that

$$\begin{aligned} \mathscr{L}\big[\, f(t) \star g(t)\,\big] &= \mathscr{L}[\, f\,]\mathscr{L}[\, g\,]\,. \\ &= F(s)G(s)\,. \end{aligned} \tag{6}$$

Proof: Let

$$\begin{aligned} I = \mathscr{L}[\, f(t) \star g(t)\,] &= \mathscr{L}\Big[\, \int_0^t f(u)g(t-u)\,du\,\Big] \\ &= \int_0^\infty e^{-st}\Big(\int_0^t f(u)g(t-u)\,du\Big)dt \\ &= \int_{t=0}^\infty \int_{u=0}^t e^{-st} f(u)g(t-u)\,du\,dt\,. \end{aligned} \tag{7}$$

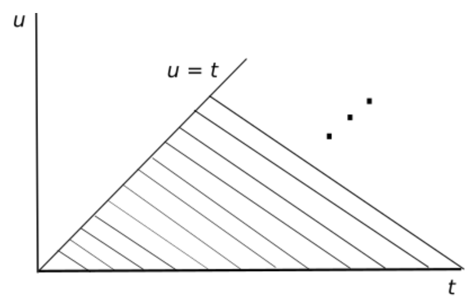

Figure 1. The region of integration will be the shaded region. We will integrate this region over vertical strips from $u = 0$ to $u = t$, and from $t = 0$ to $t = \infty$.

After changing the order of integration, we will integrate this region over from $t = u$ to $t = \infty$ and from $u = 0$ to $u = t$.

$$I = \int_{u=0}^\infty \int_{t=u}^\infty e^{su} f(u)g(t-u)\,dt\,du\,. \tag{8}$$

Then, to bring this result into its final form, we make the variable substitution: $z = t - u$. Then, at fixed u, $dz = dt$, and then

$$I = \int_{u=0}^\infty \int_{z=0}^\infty e^{-s(u+z)} f(u)g(z)\,dz\,du\,. \tag{9}$$

At this point, we can separate the integrals.

$$\begin{aligned}\mathscr{L}[\,f(t)\star g(t)\,] &= \int_{u=0}^{\infty} e^{-su} f(u)\,du \int_{z=0}^{\infty} e^{-sz} g(z)\,dz \\ &= \mathscr{L}[\,f\,]\mathscr{L}[\,g\,] \\ &= F(s)G(s)\,. \end{aligned} \tag{10}$$

3 How to exploit this theorem

Say we have the task of finding the Laplace inverse to the quantity $H(s)$, that is

$$h(t) = \mathscr{L}^{-1}[\,H(S)\,]\,, \tag{11}$$

but $H(s)$ isn't in the tables, perhaps because it's too complicated. But what if we could factor $H(s)$ into the product of two factors $F(s)$ and $G(s)$, each of which is simpler than $H(s)$? Then,

$$H(s) = F(s)G(s)\,, \tag{12}$$

where the choice of how to determine F and G is totally up to us, and it is in no way a unique factorization. And, what if each of $F(s)$ and $G(s)$ is in the table we have? In that case, we can look them up in our table:

$$f(t) = \mathscr{L}^{-1}[\,F(s)\,] \quad \text{and} \quad g(t) = \mathscr{L}^{-1}[\,G(s)\,]\,. \tag{13}$$

Great! But where does that leave us? Because of the Convolution Theorem, it leaves us with

$$h(t) = f(t)\star g(t) = \int_0^t f(u)g(t-u)\,du\,. \tag{14}$$

In other words, at this point we just convolve $f(t)$ and $g(t)$ to get the answer, though the integral may be difficult.

4 An example problem

Suppose we're given the task of finding the Laplace inverse of

$$H(s) = \frac{1}{s^2(s-a)}\,, \tag{15}$$

where a is a fixed complex number. Let's say that our small table of Laplace inverses does not have the inverse of $1/s^2(s-a)$, but does have the Laplace inverses of both $1/(s-a)$ and $1/s^2$. Then we can define

$$F(s) = \frac{1}{s^2} \quad \text{and} \quad G(s) = \frac{1}{s-a}\,. \tag{16}$$

And, from our lookup table, we have that

$$f(t) = t \quad \text{and} \quad g(t) = e^{at}\,. \tag{17}$$

Then,

$$\begin{aligned} h(t) = f(t)\star g(t) &= \int_0^t u e^{a(t-u)}\,du \\ &= e^{at}\Big[\int_0^t u e^{-au}\,du\,\Big] \\ &. \end{aligned} \tag{18}$$

Now,

$$\int ue^{-au}\,du = \left(\frac{u}{-a} - \frac{1}{a^2}\right)e^{-au}\,. \tag{19}$$

So,

$$\begin{aligned}\int_0^t ue^{-au}\,du &= \left(\frac{u}{-a} - \frac{1}{a^2}\right)e^{-au}\Big|_0^t \\ &= -\left(\frac{t}{a} + \frac{1}{a^2}\right)e^{-at} + \frac{1}{a^2}\,. \end{aligned} \tag{20}$$

Therefore,

$$h(t) = -\left(\frac{t}{a} + \frac{1}{a^2}\right) + \frac{1}{a^2}e^{at}\,. \tag{21}$$