Laplace Transform, Convolution Theorem, and the Unipodal Algebra, 1

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1 Getting Started

The unipodal algebra allows us to perform algebraic operations not as easily obtained in other algebras, owing to the fact that we can mix both imaginary numbers and unipotent numbers together in a single algebraic expression and then treat it pretty much as one would complex number for the purposes of differentiation and integration, with certain restrictions. This will be clear soon.

For the purpose of this example problem, I'm assuming that the reader knows a bit about the Laplace transform and the convolution theorem.

The unipodal algebra is formed by all linear combinations (with complex coefficients) of the two basis vectors $\{1, u\}$ (forming the *standard basis*), where u is some number not ± 1 whose square is unity, making it a *unipotent* number.

$$u^2 = 1\,. \tag{1}$$

The unipotent number commutes with complex numbers, making the unipodal algebra commutative.

And here's a handy trick: Let a and b be complex numbers, then

$$u(a + bu) = au + b\,. \tag{2}$$

An important operation on unipodal numbers is the 'selector' operation. Viewed over the reals, the unipodal algebra has four linearly independent basis elements, that being, $1, i, u, iu$. These can be selected for individually or in combination. For example, for unipodal number A with

$$A = a + bi + cu + dui\,, \tag{3}$$

where a, b, c, d are real, then

$$\langle\, A \,\rangle_{ui} = d\,, \tag{4}$$

where we stripped off the nonreal factor ui.

2 An example problem

So, here's an example problem: Find the inverse Laplace transform of

$$H(s) = \frac{ab}{(s^2 - a^2)(s^2 + b^2)}\,, \tag{5}$$

where a, b are real numbers. At this point, I'll factor $H(s)$ to

$$H(s) = F(s)G(s)\,, \tag{6}$$

where

$$F(s) = \frac{a}{s^2 - a^2} \quad \text{and} \quad G(s) = \frac{b}{s^2 + b^2}\,. \tag{7}$$

Now, we just use a Laplace transform table to lookup the inverses of $F(s)$ and $G(s)$ to get

$$f(t) = \sinh at \quad \text{and} \quad g(t) = \sin bt\,. \tag{8}$$

The Convolution Theorem proves that

$$h(t) = f(t) \star g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau\,. \tag{9}$$

In our particular case, we have

$$h(t) = (\sinh at) \star (\sin bt) = \int_0^t (\sinh a\tau)\sin b(t-\tau)\,d\tau\,. \tag{10}$$

3 Computing the convolution

The way I'll solve this is to use a trick that's well-know in the integration of circular trigonometric functions, which is to employ the polar form of the trig functions when convenient to do so. The reason to do this is that the polar forms are often easiler to integrate.

So, consider the product

$$e^{a\tau u}e^{b(t-\tau)i} = (\cosh a\tau + u\sinh a\tau)(\cos b(t-\tau) + i\sin b(t-\tau))\,. \tag{11}$$

If we expand this out, we see that the product we want is the ui term. Hence,

$$\sinh a\tau \sin b(t-\tau) = \langle\, e^{a\tau u}e^{b(t-\tau)i}\,\rangle_{ui}\,. \tag{12}$$

So, going back to (10), we get that

$$
\begin{aligned}
h(t) &= \int_0^t (\sinh a\tau)\,\sin b(t-\tau)\,d\tau \\
&= \int_0^t \langle\, e^{a\tau u} e^{b(t-\tau)i}\,\rangle_{ui}\,d\tau \\
&= \langle \int_0^t e^{a\tau u} e^{b(t-\tau)i}\,d\tau\,\rangle_{ui} \\
&= \langle \int_0^t e^{bti+(au-bi)\tau}\,d\tau\,\rangle_{ui}\,. \qquad (13)
\end{aligned}
$$

Now, since the integration does not range over the variable t, we can pull functions of t out from the integrand, as such

$$
h(t) = \langle\, e^{bti} \int_0^t e^{(au-bi)\tau}\,d\tau\,\rangle_{ui}\,. \qquad (14)
$$

Conceptually, the way is straigthforward, but there will be difficulties in the details. Next, we will evaluate only the integral part of (14).

$$
\begin{aligned}
\int_0^t e^{(au-bi)\tau}\,d\tau &= \left.\frac{e^{(au-bi)\tau}}{au-bi}\right|_0^t \\
&= \frac{e^{(au-bi)t}-1}{au-bi} \\
&= \frac{u}{u}\,\frac{e^{(au-bi)t}-1}{au-bi} \\
&= u\,\frac{e^{(au-bi)t}-1}{a-biu}\,. \qquad (15)
\end{aligned}
$$

But iu is just an imaginary number inasmuch as $(iu)^2 = -1$. Therefore

$$
\begin{aligned}
\int_0^t e^{(au-bi)\tau}\,d\tau &= u\,\frac{e^{(au-bi)t}-1}{a-biu}\,\frac{a+biu}{a+biu} \\
&= u\,\frac{[e^{(au-bi)t}-1](a+biu)}{a^2+b^2} \\
&= \frac{1}{a^2+b^2}[e^{(au-bi)t}-1](au+bi)\,. \qquad (16)
\end{aligned}
$$

Now, if we multiply through by e^{bti}, we get

$$
\begin{aligned}
e^{bti}\int_0^t e^{(au-bi)\tau}\,d\tau &= \frac{1}{a^2+b^2}[e^{aut}-e^{bti}](au+bi) \\
&= \frac{1}{a^2+b^2}[(\cosh at + u\sinh at) - (\cos bt + i\sin bt)](au+bi)\,. \qquad (17)
\end{aligned}
$$

Lastly, we extract from the RHS the ui terms:

$$h(t) = \frac{1}{a^2+b^2}[\, b \sinh at - a \sin bt \,]\,. \tag{18}$$

4 Confirming the answer

If (18) is correct, we should be able to confirm it easily enough. Is

$$\mathscr{L}[\, h(t) \,] = \frac{ab}{(s^2-a^2)(s^2+b^2)}\ ? \tag{19}$$

Well, let's see.

$$\begin{aligned} \mathscr{L}[\, h(t) \,] &= \int_0^\infty \frac{1}{a^2+b^2} e^{-st}[\, b \sinh at - a \sin bt \,]\, dt \\ &= \frac{b}{a^2+b^2} \int_0^\infty e^{-st} \sinh at\, dt - \frac{a}{a^2+b^2} \int_0^\infty e^{-st} \sin bt\, dt\,. \end{aligned} \tag{20}$$

Let's do each integral separately.

$$\int_0^\infty e^{-st} \sinh at\, dt = \frac{e^{-st}}{s^2-a^2}[\, a \cosh at - s \sinh at \,]\Big|_{t=0}^{\infty} = \frac{a}{s^2-a^2}\,, \tag{21}$$

and

$$\int_0^\infty e^{-st} \sin bt\, dt = \frac{e^{-st}}{s^2+b^2}[\, -s \sin bt - b \cos st \,]\Big|_{t=0}^{\infty} = \frac{b}{s^2+b^2}\,. \tag{22}$$

On substituting these into (20), we get

$$\begin{aligned} \mathscr{L}[\, h(t) \,] &= \frac{b}{a^2+b^2}\,\frac{a}{s^2-a^2} - \frac{a}{a^2+b^2}\,\frac{b}{s^2+b^2} \\ &= \frac{ab}{a^2+b^2}\left(\frac{1}{s^2-a^2} - \frac{1}{s^2+b^2}\right) \\ &= \frac{ab}{(s^2-a^2)(s^2+b^2)}\,. \end{aligned} \tag{23}$$

So, if I did the math correctly, it is confirmed.