

The Lambert W Function

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Abstract

The Lambert W function is not well-know. Although it's not widely applicable in mathematics (with some applications in physics and other sciences), where it does apply, it is the only hope to get a closed-form solution to certain algebraic equations.

“Small moves, Ellie. Small moves.”
— Ellie’s father (*Contact*)

1 Introduction

The Lambert W function is that function that satisfies the following equation¹

$$xe^x = z, \quad (1)$$

where z is given. Thus, with use of the Lambert W function, this last equation has solution

$$W(z)e^{W(z)} = z. \quad (2)$$

Since I intend this paper to be a very short introduction to the Lambert W function, I will not go into much detail about it, except to say that in a real variable, it has domain $[-\frac{1}{e}, \infty)$. And on this domain, W is invertible, hence $W^{-1}(z)$ exists.

This brings us to our first lemma: Given (2), show that $ze^z = W^{-1}(z)$.

We begin with

$$W(z)e^{W(z)} = z. \quad (3)$$

Next, we introduce the temporary variable y as

$$y = W(z). \quad (4)$$

Therefore, because W is invertible,

$$z = W^{-1}(y). \quad (5)$$

¹The Lambert W function is accessed in WolframAlpha as ‘productlog()’.

Substituting this into (3), we get

$$W(W^{-1}(y))e^{W(W^{-1}(y))} = z. \quad (6)$$

Using the fact that

$$W(W^{-1}(y)) = y, \quad (7)$$

we get for (6),

$$ye^y = z. \quad (8)$$

But from (5) we get, by substitution, that

$$ye^y = W^{-1}(y). \quad (9)$$

Now, on replacing the dummy variable y by z , we get

$$ze^z = W^{-1}(z). \quad (10)$$

And, since W is invertible, we can write this in the alternative form as

$$W(ze^z) = z. \quad (11)$$

2 Some Problems to Solve

Problem 1) Solve for x in

$$x^{\alpha x} = b. \quad (12)$$

Solution: Take the natural log across this equation then divide by α :

$$x \ln x = \frac{\ln b}{\alpha} = c. \quad (13)$$

Next, let

$$x = e^y \quad (14)$$

and substitute this into (13), to get

$$e^y \ln e^y = c, \quad (15)$$

which simplifies to

$$ye^y = c. \quad (16)$$

Take the W function across this equation and get

$$W(ye^y) = W(c) = y. \quad (17)$$

Hence,

$$x = e^y = e^{W(c)} = \exp \left[W \left(\frac{\ln b}{\alpha} \right) \right]. \quad (18)$$

Problem 2) Solve for x in

$$x^n = a^x, \quad (19)$$

where n is an integer and a is a positive real number.

Solution: Take the natural log across this equation:

$$n \ln x = x \ln a. \quad (20)$$

Next, replace x by y^{-1} :

$$-n \ln y = y^{-1} \ln a. \quad (21)$$

After a little manipulation, we get

$$y \ln y = -\frac{\ln a}{n}. \quad (22)$$

Next, we let $y = e^z$:

$$ze^z = -\frac{\ln a}{n} = c. \quad (23)$$

On taking W across this equation, we get

$$W(ze^z) = W(c) = z. \quad (24)$$

Hence,

$$y = e^z = e^{W(c)} = \exp \left[W \left(-\frac{\ln a}{n} \right) \right]. \quad (25)$$

And, finally,

$$x = y^{-1} = \exp \left[-W \left(-\frac{\ln a}{n} \right) \right]. \quad (26)$$

Problem 3) Solve for x in

$$2^x = ax, \quad (27)$$

where a is a positive real number.

Solution: Replace x by y/a , to get

$$2^{y/a} = y, \quad (28)$$

then take the natural log across this equation:

$$\frac{y}{a} \ln 2 = \ln y. \quad (29)$$

Next, replace y by z^{-1} :

$$\frac{z^{-1}}{a} \ln 2 = -\ln z. \quad (30)$$

After a little manipulation, we get

$$z \ln z = -\frac{\ln 2}{a} = c. \quad (31)$$

Next, we let $z = e^u$:

$$ue^u = -\frac{\ln 2}{a} = c. \quad (32)$$

On taking W across this equation, we get

$$W(ue^u) = W(c) = u. \quad (33)$$

Hence,

$$z = e^u = e^{W(c)} = \exp\left[W\left(-\frac{\ln 2}{a}\right)\right]. \quad (34)$$

Then

$$y = \exp\left[-W\left(-\frac{\ln 2}{a}\right)\right]. \quad (35)$$

And, finally,

$$x = \frac{1}{a} \exp\left[-W\left(-\frac{\ln 2}{a}\right)\right]. \quad (36)$$

Problem 4) Solve for x in

$$x + e^x = 2. \quad (37)$$

I got this problem off of a YouTube video by Polar Pi, called “Solving Equations with the Lambert W function”:

https://www.youtube.com/watch?v=F_P1atH3cZk

First, I’ll show the solution presented there, and after that, I’ll show a different solution. Then comes a surprise!

First Solution: Subtract x from both sides, to get

$$e^x = 2 - x. \quad (38)$$

Multiply through by e^{-x} :

$$1 = (2 - x)e^{-x}. \quad (39)$$

Next, multiply through by e^2 :

$$e^2 = (2 - x)e^{2-x}. \quad (40)$$

On taking W across this equation, we get

$$W(e^2) = W((2 - x)e^{2-x}) = 2 - x. \quad (41)$$

Hence,

$$x = 2 - W(e^2). \quad (42)$$

Second Solution: Let $y = e^x$ and substitute:

$$\ln y + y = 2. \quad (43)$$

Then, use these as exponents:

$$e^{[\ln y + y]} = ye^y = e^2. \quad (44)$$

On taking W across this equation, we get

$$W(ye^y) = y = W(e^2). \quad (45)$$

Hence,

$$x = \ln y = \ln [W(e^2)]. \quad (46)$$

Therefore, I have to wonder if

$$\ln [W(e^2)] = 2 - W(e^2). \quad (47)$$

Well, I was pretty sure they had to be equal, but I didn't know how to prove it. So, with nothing to lose by asking BingChat, I asked it to prove that

$$\ln [W(e^2)] = 2 - W(e^2). \quad (48)$$

The following is the conversation I had with BingChat:

I asked BingChat to prove that: $2 - W(e^2) = \ln(W(e^2))$ where W is the Lambert W function.

This is what it replied:

We can start by using the definition of the Lambert W function:

$$W(z)e^{W(z)} = z. \quad (49)$$

Substituting $z = e^2$, we have:

$$W(e^2)e^{W(e^2)} = e^2. \quad (50)$$

Taking the natural logarithm of both sides, we get:

$$\ln(W(e^2)) + W(e^2) = 2. \quad (51)$$

Subtracting $W(e^2)$ from both sides, we have:

$$\ln(W(e^2)) = 2 - W(e^2). \quad (52)$$

Therefore,

$$2 - W(e^2) = \ln(W(e^2)). \quad (53)$$

Bravo, BingChat!

References

- [1] T. P. Dence, “A Brief Look into the Lambert W Function”, *Applied Mathematics*, (2013), 887–892.

https://www.scirp.org/pdf/AM_2013060409554653.pdf