Surjective (onto) Functions

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Abstract

There are times when one needs to know if a given function is onto (surjective) or one-to-one (injective). In this article, we will investigate surjective functions.

"Small moves, Ellie. Small moves."
— Ellie's father (*Contact*)

1 Introduction

For our purposes in this paper, a *function* is a way to map the elements of one set to elements of another set. The first or starting set is referred to as the *domain*, and the second set is referred to as the *codomain*. For our purposes, the domain and codomain are nonempty sets, meaning that each set contains at least one element. An element of the codomain that gets mapped to by some element of the domain is said to be an image point (or element) of the function.

The set of all image points in the codomain is called the *image* of the function. An element in the domain that gets mapped to a given element of the codomain is said to be a *pre-image* of that codomain element. Since more than one domain element can be mapped to the same element in the codomain, the preimage of a given codomain element may be one or many elements of the domain. Of course, this preimage set is a subset of the domain. If the function is 1-1, the preimage sets of each codomain point contains a single element.

Finally, unless stated otherwise, functions are considered not to be multivalued, meaning that the function maps each domain element to exactly one codomain element.

There are three basic kinds of image sets that are of particular importance in set theory and in higher mathematics that are used to classify functions. They are 1) Functions whose image sets whose every element is the image of exactly one domain point. Such a function is said to be *injective* (1-1). In other words, a function is injective if there are never two distinct elements of the domain that are mapped to the same element of the codomain. 2) Functions whose image sets cover all of the codomain are said to be *surjective* or *onto*.

Surjective functions may or may not be 1-1. 3) Functions whose image sets are both surjective and injective, are called *bijective*.

Now, I want to define a term that remedies what I believe is an obvious omission in the literature: A function is said to be *enjective* (or *into*) if it is not surjective. In other words, a function is said to be enjective is there is some codomain point that has no preimage under the mapping.

There's a property of functions that is easy to forget at the very moment you need to remember it, and that is that under the action of a function, every element of the domain is mapped to some element of the codomain.

Definition: The *cardinality* of a set is the number of elements of the set. For set S, its cardinality would be represented as $|\, S\,|$.

2 Easy Cases of Surjective Functions

Problem 1) Let $S = \{a, b, c\}$ and $T = \{A, B, C\}$. Let f be a function from S to T, given by the rule that the elements of S are mapped to their uppercase versions in T. Clearly the function is 1-1 and clearly it's also surjective, because every element of T has a preimage in S.

Problem 2) Let $S = \{0, 1, 2, \ldots\}$, that is, the set of nonnegative integers. And let $T = \{0, 1\}$. Let f be a function from S to T, given by the rule that $\forall\, a \in A$,

$$f(a) = \text{remainder } a/2\,. \tag{1}$$

So, when a is even it gets mapped to 0; and when it's odd it gets mapped to 1. Therefore, each element of the codomain has at least one preimage point in S (in fact, each has an infinite number), making the function f surjective. But what happens if we change set T to $T' = \{0, 1, 2\}$ under the same mapping? Well, even elements still get mapped to 0 and odd elements still get mapped to 1, as before, but no element will get mapped to the element 2; therefore, the mapping is enjective, not surjective.

Now, I'll state, without proof, an obvious lemma:

Lemma 1

Let S and T be finite sets. Then, if f is a function from S to T, and the cardinality of S is less than the cardinality of T (or $|\, S\,| < |\, T\,|$), then the function is enjective.

Lemma 2

A function is always a surjective map to its image set, when it's image set is treated as the function's codomain.

3 More Interesting Cases of Surjective Functions

Problem 3) Let f be a function from the x-axis ($S = \mathbb{R}$) to the y-axis ($T = \mathbb{R}$) in the real plane, given by

$$y = f(x) = mx + b\,, \tag{2}$$

where m is finite (eliminating a vertical line case). By the way, for the case $m = 0$, we have the situation where all the real numbers in S get mapped to a single y value in the codomain T ($y = b$), and this function is definitely not surjective.

So, with the added restriction that $m \neq 0$, if we pick an arbitrary y value, say y_0, can we solve for some x_0 that gets mapped to it under the function f? In other words, does an arbitrary y number have a preimage in S? If we can prove this by finding a formula of the form $x_0 = x_0(y_0)$ then we are done—so long as this x_0 is actually in the domain. In order to accomplish this, we must take Eq. (2) and turn it inside out.

$$x_0 = x(y_0) = \frac{y_0 - b}{m}\,. \tag{3}$$

As a test,

$$f(x_0) = m\big(\frac{y_0 - b}{m}\big) + b = y_0\,. \tag{4}$$

This last equation does indeed prove that we found at least one $x \in S$, namely $x = x_0$, that gets mapped to y_0 under the function f. Therefore, every $y \in T$ has a preimage in S. Thus, f is surjective.

Lemma 3

An injective function f between finite sets S and T of the same cardinality is necessarily surjective, and thus also bijective.

Proof:

Let's call the cardinality of the two sets to be n. This proof will be by contradiction. That is, we'll assume that the conclusion is not true and then get a contradiction with the given information. If we get this contradiction, then our initial assumption that the conclusion is false is itself false.

For simplicity, we assume that there is exactly one point in the codomain that has no preimage. We'll call it α. That leaves $n-1$ elements in S whose images in T are the remaining $n-1$ points in T, paired 1-1. Let a be the point in S that we haven't yet assigned an image in T. Now, since f maps every point is S to some point in T, what point (element) does that leave for $f(a) \in T$? It can't be to α or that would force f to be surjective. But it also can't be to any other element of T because they already have preimages, and if it went to one of them, then that element would have more than one preimage element, making f non-injective — another contradiction. Our only choice left is to admit that our original assumption that f is not surjective is false, hence, f is surjective.

Lemma 4 Let S and T both be infinite sets. Then, if f is an injection from S to T, it's not always true that f is surjective. In other words, we cannot generalize Lemma 3 to sets of infinite cardinality, not even if S and T are the same sets. The claim here is not that f is never a surjection, but that we can find cases where it isn't.

First, let's present a case where it does hold. Let S and T both be the integers. Let f be the function, such that

$$f(x) = x\,. \tag{5}$$

It's easy to prove that this function is 1-1. Assume that $f(a) = f(b)$ for a pair of domain elements a, b. Then,

$$f(a) = f(b)\,. \tag{6}$$

Then, using (5), we get that

$$a = b\,. \tag{7}$$

Since $a = b$, these are not distinct points, so f is injective. But it's also surjective because for an arbitrary $y_0 \in T$, we choose $x_0 = y_0$. But since S and T are the same sets, clearly the existence of $y_0 \in S$ is guaranteed.

Problem 4) Let f be a function given by

$$f : D \to D, \quad \text{where} \quad D = (1, \infty)\,, \tag{8}$$

and

$$f(x) = \frac{1}{x-1}\,. \tag{9}$$

Is f surjective?

So, if we pick an arbitrary y_0 from T, can we determine an x_0 from S that f maps to it? Let's try by beginning with the assumption that it is possible. Then

$$y_0 = f(x_0) = \frac{1}{x_0 - 1}\,, \tag{10}$$

can be solved for x_0 to get

$$x_0 = \frac{1}{y_0} + 1\,. \tag{11}$$

First, we'll test that this point x_0 is mapped to y_0:

$$f(x_0) = \frac{1}{\left[\frac{1}{y_0} + 1\right] - 1} = y_0\,, \tag{12}$$

so that checks out. But so far we have only assumed that this point x_0 is in the domain. Now we have to prove it. The function $\frac{1}{y_0} + 1$ from (11) is defined on $D = (1, \infty)$. As $y_0 \to 1$, $x_0 \to 2$, which is consistent with the domain. As

$y_0 \to \infty$, $x_0 \to 1$ from above, which is also consistent with the domain. Thus, This f is surjective.

Problem 5) Let f be a function given by

$$f : \mathbb{R}^+ \to \mathbb{R}^+ \,, \tag{13}$$

and

$$f(x) = x^2 + 2x \,. \tag{14}$$

Show that f is surjective. Our task now is to find some x_0 that solves the equation

$$y_0 = x_0^2 + 2x_0 \,, \tag{15}$$

which can be rewritten in standard quadratic form as

$$x_0^2 + 2x_0 - y_0 = 0 \,, \tag{16}$$

where y_0 is assumed to be given, of course. On using the quadratic formula, we get

$$\begin{aligned} x_0 &= \frac{-2 \pm \sqrt{4 - 4(1)(-y_0)}}{2} \\ &= -1 \pm \sqrt{1 + y_0} \,. \end{aligned} \tag{17}$$

And, by choosing to use the plus sign of the square root, we will always find an x_0 in the domain of f.

Theorem: Show that the composition of surjective functions is surjective.

Direct proof:

Let $f : A \to B$ be a surjective function from set A to set B. Further, let $g : B \to C$ be a surjective function from set B to set C. To show that the composition of f and g, namely $g \circ f$ is also surjective, we need to show that for an arbitrary element γ of C, we can find some element α of A, such that $(g \circ f)(\alpha) = \gamma$.

So, we begin with the fact that for an arbitrary $\gamma \in C$, there exists some $\beta \in B$ (because g is surjective), such that

$$g(\beta) = \gamma \,. \tag{18}$$

But, because f is surjective, there exists some $\alpha \in A$ such that

$$f(\alpha) = \beta \,. \tag{19}$$

So, upon substituting this last equation into the one before it, we get that

$$g(f(\alpha)) = (g \circ f)(\alpha) = \gamma \,. \tag{20}$$

What we have proven is that for an arbitrary $\gamma \in C$ there exists some $\alpha \in A$ such that $(g \circ f)(\alpha) = \gamma$, which means that the composition of these two surjective functions is itself surjective.