Example of Generating Functions Applied to Special Relativity

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February 25, 2020

Abstract

Here is a clever application of generating functions applied to the velocity addition formula in special relativity, when applied recursively.

1 Introduction

Although this paper is designed to be read by physicists and physics students to learn how to use power series generating functions in ad hoc situations, the paper should be accessible to mathematics students, as well.

Generating functions in physics come in a variety of forms besides power series, such as generating functions used in Hamiltonian mechanics. The routine application of power series generating functions that physics students encounter is likely to be for use in so-called special functions, such as in the study of Legendre and Chebyshev polynomials. But our use of generating functions in this paper is an example of how to employ them as the need arises spontaneously in one's study of physics.

From here on, when the phrase 'generating function' is used, it will refer to a 'power series generating function'. We assume that the reader already knows a little about (formal) power series. The importance of our power series being 'formal' is that we do not concern ourselves with questions of convergence.

This paper presents and expands on a problem concerning generating functions as applied to a problem in special relativity, as I found in the paper, "Generating Functions and Their Applications," by Agustinus Peter Sahanggamu, found on page 2 of

`https://ocw.mit.edu/courses/mathematics/18-104-seminar-in-analysis -applications-to-number-theory-fall-2006/ projects/peter_s.pdf`

On page 1, the author says that, "A generating function is a continuous function associated with a given sequence." My own take on this is that *a generating function is a means to encode an infinite sequence of numbers or functions with a rational polynomial function.*

Suppose we have an infinite sequence of functions $\{f_n\}$, where n goes from 0 to ∞, and where the f_n satisfy a linear recurrence relation of the form

$$f_{n+1} = af_n + b\,, \tag{1}$$

where f_0, a, and b are given constants, i.e., not dependent on n. What we seek is a closed form for the unknown functions f_n. I call the f_n 'functions' because they are dependent on n, f_0, a, and b:

$$f_n = f_n(n, f_0, a, b)\,. \tag{2}$$

Then, by the method of generating functions, we seek to find a closed form for these functions f_n by inventing the generating function $f(x)$ defined by

$$f(x) = \sum_{n=0}^{\infty} f_n x^n\,, \tag{3}$$

in the hope that, combined with the information in (1), we can find a rational function $G(f(x), f_0, a, b)$ such that

$$G(f(x), f_0, a, b) = 0\,, \tag{4}$$

by which we can 1) algebraically solve for $f(x)$ as a simple rational function of x and parameters f_0, a, and b, and 2) express $f(x) = f(x, n, f_0, a, b)$ in terms of power series as

$$f(x) = \sum_{n=0}^{\infty} F_n x^n\,. \tag{5}$$

Taking this last equation together with (3), we can find a closed form for f_n by setting $f_n = F_n$, term by term.

In other words, the information contained in (1) is used to find (4), which is used to algebraically solve for $f(x, n, f_0, a, b)$ as a rational function of its arguments, which is then expressed as a power series, thereby giving us a closed form for $f_n = F_n(n, f_0, a, b)$. For the generic case posed above, we get the formula:

$$f_n = \Big(f_0 - \frac{b}{1-a}\Big)a^n + \frac{b}{1-a}\,. \tag{6}$$

The reader is encouraged to try to prove (6) before continuing. In any case, the proof is very similar to the proof I use below as I follow rather closely the author's proof.

2 Working the problem

Now, on to the problem. (Note: The equation numbers are relative to *this* paper, not the author's paper:

Example 2-3. In special relativity, the usual one-dimensional velocity addition formula $v' = u + v$ is modified into [1, p. 127]

$$v' = \frac{u+v}{1+uv} \tag{7}$$

with v, u, and v measured in units of the speed of light c. We will use this velocity addition in the following problem. Suppose there are infinitely many cars labeled by integers $n \geq 0$. The $(n+1)$-th car moves to the right relative to the n-th car with a relative velocity v $(0 < v < 1)$. In our reference frame, we denote the velocity of the n-th car by u_n. Assuming $u_0 = 0$, find u_n for all $n \geq 1$.

■

Now, for my solution, which is based on the author's solution, with the main exception being that I will define the reference frames and their velocities a bit differently. The situation for my restatement of the problem is posed in Figure 1. Note: The speed u_m is the speed of particle P in some frame S_m, and that speed will vary depending on which frame is making the measurement or calculation, as the situation may be.

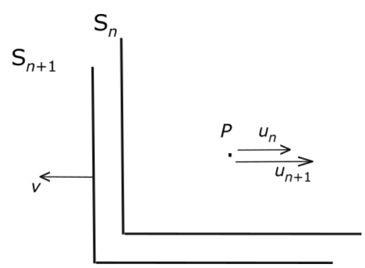

Figure 1. Here I use a standard relative-frame depiction of the motion of the point particle P. All motion is along the horizonal axis. All velocity vectors are relative to the S_n frame, except velocity u_{n+1}, which is relative to S_{n+1} frame.

First, I want to treat u and v as nonnegative speeds, and define the velocities from there. We begin with frame S_0, which is the rest frame of point particle P, hence $u_0 = 0$. Then we imagine the nth reference frame S_n in which P moves in the positive x-direction with velocity $+u_n$. Next, we imagine a boost by speed v from frame S_n to frame S_{n+1} in the negative x direction (where frame S_{n+1} is the primed frame). The speed of P in S_{n+1} is $u_{n+1} > u_n$, and given by the Einstein velocity addition formula (7) above, with u and v interchanged:

$$u_{n+1} = \frac{u_n + v}{1 + u_n v}\,. \tag{8}$$

The next step is to use algebra to convert (8) into the form:

$$f_{n+1} = af_n + b\,, \tag{9}$$

where f_n is a function of u_n, and the constants a and b. By use of some clever algebraic manipulations, we get (using the author's notation)

$$f_{n+1} = \alpha f_n - \lambda\,, \tag{10a}$$

where

$$f_n = \frac{1}{1-u_n}\,, \quad \alpha = \frac{1+v}{1-v}\,, \quad \lambda = \frac{v}{1-v}\,. \tag{10b}$$

Let's work out how this transformation is done. We begin with a trivial identity and then use (8):

$$\begin{aligned} 1 - u_{n+1} &= 1 - \frac{u_n + v}{1 + u_n v} \\ &= \frac{1 + u_n v}{1 + u_n v} - \frac{u_n + v}{1 + u_n v} \\ &= \frac{1 + u_n v - (u_n + v)}{1 + u_n v} \\ &= \frac{(1-u_n)(1-v)}{1 + u_n v}\,. \end{aligned} \tag{11}$$

Next, we invert both sides and expand the RHS:

$$\begin{aligned} \frac{1}{1-u_{n+1}} &= \frac{1 + u_n v}{(1-u_n)(1-v)} \\ &= \frac{1 + v - v(1-u_n)}{(1-u_n)(1-v)} \\ &= \frac{1+v}{(1-u_n)(1-v)} - \frac{v(1-u_n)}{(1-u_n)(1-v)} \\ &= \frac{1+v}{1-v}\frac{1}{1-u_n} - \frac{v}{1-v} \\ &= \alpha\frac{1}{1-u_n} - \lambda\,. \end{aligned} \tag{12}$$

From this last equation and (10b), we get (10a).

But how to proceed from here? There are some standard tricks to apply. One is the use of the geometric series put into rational form

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\,. \tag{13}$$

Another is to take our linear recurrence relation (10a) and to 'sum on it' across the equation, thusly,

$$\sum_0^{\infty} f_{n+1}x^n = \alpha\sum_{n=0}^{\infty} f_n x^n - \lambda\sum_0^{\infty} x^n\,, \tag{14}$$

the point of which is to transform the information in (10a) into a rational function of $f(x)$. Making some simple changes, we get

$$\sum_{n=0}^{\infty} f_{n+1} x^n - \alpha f(x) + \frac{\lambda}{1-x} = 0\,. \tag{15}$$

But what to do with $\sum_{n=0}^{\infty} f_{n+1} x^n$? We must manipulate this summation to get something of the form $\sum_{n=0}^{\infty} f_n x^n$, which can then be replaced by $f(x)$. This is accomplished by 'index shifting':

$$\begin{aligned}
\sum_{n=0}^{\infty} f_{n+1} x^n &= \frac{1}{x} \sum_{n=0}^{\infty} f_{n+1} x^{n+1} \quad \text{(the setup: matchup indices/exponents)} \\
&= \frac{1}{x} \sum_{n=1}^{\infty} f_n x^n \quad \text{(make the index shift: } n \to n-1\text{)} \\
&= \frac{1}{x} \Big[\sum_{n=0}^{\infty} f_n x^n - f_0 \Big] \quad \text{(make the summation start at 0)}\,,
\end{aligned} \tag{16}$$

where, in the last step we added in $f_0 x^0$ and subtracted out f_0, which, by the way, is equal to 1.

Therefore, (15) becomes

$$\frac{1}{x}[f(x) - 1] - \alpha f(x) + \frac{\lambda}{1-x} = 0\,, \tag{17}$$

and this equation corresponds to (4). Solving this last equation for $f(x)$ yields

$$f(x) = \frac{1}{2}\Big[\frac{1}{1-x} + \frac{1}{1-\alpha x}\Big]\,, \tag{18}$$

where we used that $\lambda/(\alpha - 1) = 1/2$.

Using (13), we convert $f(x)$ into a power series to conform to (5), yielding,

$$f(x) = \frac{1}{2}\Big[\sum_{0}^{\infty} x^n + \sum_{0}^{\infty} \alpha^n x^n\Big] = \sum_{n=0}^{\infty} [\tfrac{1}{2}(1+\alpha^n)] x^n = \sum_{n=0}^{\infty} F_n x^n\,. \tag{19}$$

Using (5) and (10b) for f_n, and that $f_n = F_n$, we get

$$\frac{1}{1-u_n} = \tfrac{1}{2}(1+\alpha^n)\,, \tag{20}$$

from which we derive

$$u_n = 1 - \frac{2}{\alpha^n + 1} = \frac{\alpha^n - 1}{\alpha^n + 1}\,, \tag{21}$$

which is our sought-after closed form.

Now, since $\alpha > 0$, then, in the limit as $n \to \infty$, $u_n \to 1$, which corresponds to the actual speed of c. Let's restate what this means: Since u_n is the speed of frame S_n with respect to particle P,[1] or its rest frame S_0, then no matter how many boosts of speed v one makes, one cannot even attain a relative speed of c, much less go beyond that. This result contradicts Galilean relativity, which has no upper limit on relative speeds between reference frames.

Let's pose a particular numerical problem at this point:

> How many iterations of boost $v = 0.05$ are required to produce a relative speed of a 0.5 with respect to the rest frame of P, i.e., S_0? In other words, What is the smallest value of n required so that u_n in frame S_n is at least to 0.5?

Ans: To answer this, we setup an inequality from (21):

$$u_n = 1 - \frac{2}{\alpha^n + 1} \geq 0.5\,. \tag{22}$$

Solving this equation for n as a real number (using algebra and logarithms), I get $n \approx 10.9733$. On treating n once again as a natural number, the number of iterations of a boost by $v = 0.05$ is 11. By the way, in Galilean relativity, one would require only 10 boosts by the same amount.

3 Conclusion

I have a vague memory of wondering how to solve the problem posed in this paper back when I was taking special relativity as a physics undergrad. But I had no knowledge of generating functions in those days, and so I left the problem unsolved for myself for all these decades. It's a joy to finally see the solution.

[1] I'm careful to use the word 'speed' here to distinguish it from velocity. The velocity of S_n with respect to S_0 is $-u_n$, as S_n moves in the negative x direction of the S_0 frame.