Much Ado About Ellipses, Part 4: Kepler's Laws

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April 26, 2022

1 Introduction

This time I'm going over **Kepler's Three Laws of Planetary Motion** because they're all related to ellipses, and this is how:

1. Kepler's First Law: The orbits of the planets around the Sun are ellipses with the Sun at one focus of the ellipse.
2. Kepler's Second Law: A radial line connecting a planet to the Sun sweeps out equal areas in equal times.
3. Kepler's Third Law: The square of a planet's period is proportional to the cube of the length of its orbit's semimajor axis.

Now we're going to prove these three laws of planetary motion by assuming Newtonian mechanics and making the approximation that the Sun is 'fixed' in space, and that the planets obey Newton's Law of gravitational attraction, and that the gravitational interaction of the planets among themselves is negligible.[1]

Now we come to **Newton's Three Laws of motion:**

1. Newton's First Law: An object at rest or in intertial motion will tend to remain in that state of motion unless acted upon by an unbalanced external force.
2. Newton's Second Law: The acceleration of an object of fixed mass m is given by the equation: $\mathbf{a} = \mathbf{F}/m$, where $\mathbf{F}$ is the sum of the forces on the object at any instant.[2] Or, as is more commonly expressed

$$\mathbf{F} = m\mathbf{a}\,. \tag{1}$$

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[1] Now, I'm not saying that in fact the planets do not interact gravitationally. I'm just saying that our analysis here is neglecting those interactions as small effects. The result we get under this assumption is a first-order approximation to the planetary orbits.

[2] The underlying assumption of this law is that it is being observed by someone in an inertial reference frame. For our purposes, one such notion of an intertial reference frame is one with constant speed and without rotation with respect to the fixed background stars.

3. Newton's Third Law: Every application of a force of a first object onto a second object engenders an equal and oppositely directed force from the second object onto the first. (Which object you label as the first and the second is irrelevant.)

In addition to these three generic laws of motion, Newton added his Law of Universal Gravitation that is assumed to exist between any two mass objects:

$$\mathbf{F} = \frac{GMm}{r^2}\hat{\mathbf{r}}\,, \tag{2}$$

where M is the mass of one object (we'll take it to be the mass of the Sun in this case), m is the mass of the other object, which will be the mass of one of the planets, r is the radial distance between the Sun and the planet, $\hat{\mathbf{r}}$ is the unit outward radial vector from the Sun, and G is Newton's universal gravitational constant.

2 Radial Forces and Angular Momentum

Our vector equation (**??**) is valid in all coordinate systems. But in this article, we will be interested in those forces that are called *radial*, meaning that the force experienced on the particle is always in line with the radial line connecting the two masses. The force can be toward the force mass (attractive force) or outward from the force mass (repellant force).

Such a force is also said to be a central force. The important thing for us is that when a central force is described in polar coordinates, then the equations take on a simple form and it's apparent that there is an important constant of the motion. We'll see this soon enough.

The next new concept to introduce is that of *angular momentum* $\mathbf{L}$.

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}\,, \tag{3}$$

where $\mathbf{r}$ is the vector from the origin to the mass point Then $\mathbf{p}$ is referred to as the linear momentum of the mass point, and is defined as

$$\mathbf{p} \equiv m\mathbf{v} = m\dot{\mathbf{r}}\,. \tag{4}$$

In any coordinate system we can write the acceleration as $\ddot{\mathbf{r}}$, then (1) becomes

$$\mathbf{F} = m\ddot{\mathbf{r}} = D_t m\dot{\mathbf{r}} = D_t\mathbf{p} = \dot{\mathbf{p}}\,, \tag{5}$$

where D_t is the derivative with respect to time.

Now, on differentiating the vector angular momentum in (3), we get

$$\begin{aligned}\dot{\mathbf{L}} &= \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \\ &= m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \mathbf{F} \\ &= 0\,. \end{aligned} \tag{6}$$

I'll explain. The vector $\dot{\mathbf{r}} \times \dot{\mathbf{r}}$ is zero because the cross product of any vector with itself is zero. The vector $\mathbf{r} \times \mathbf{F}$ is zero for a similar reason (use Eq. (2)). Namely, $\mathbf{r} \times \hat{\mathbf{r}} = 0$.

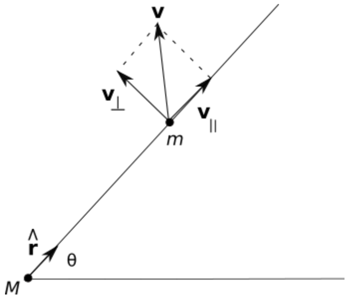

Figure 1. The point at bottom left represents the origin of the polar coordinate system.

The *angular speed* of the point mass is defined as

$$\omega \equiv \dot{\theta}\,. \tag{7}$$

The magnitude of the velocity perpendicular to the radial vector at the moving mass point is given by

$$v_\perp = r\omega\,. \tag{8}$$

But

$$v_\perp = |\,\mathbf{v}\,|\sin\theta\,. \tag{9}$$

Hence, from these last two equations we can write

$$\omega = \frac{|\,\mathbf{v}\,|\sin\theta}{|\,\mathbf{r}\,|} = \frac{v\sin\theta}{r}\,. \tag{10}$$

We can build the vector angular velocity by

$$\boldsymbol{\omega} = \frac{\mathbf{r}\times\mathbf{v}}{r^2}\,, \tag{11}$$

where $\boldsymbol{\omega}$ is in the $\boldsymbol{\sigma}_3$ direction, and

$$\mathbf{v}_\perp = \boldsymbol{\omega}\times\mathbf{r}\,. \tag{12}$$

Now, we can superimpose a cartesian coordinate system with origin at the fixed mass and project our vectors onto the standard basis of $\boldsymbol{\sigma}_1$ in the positive x direction and $\boldsymbol{\sigma}_2$ in the positive y direction. Then, with

$$\hat{\mathbf{r}} = \mathbf{r}/r\,, \tag{13}$$

we get

$$\hat{\mathbf{r}} = \frac{x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2}{r} = \frac{r\cos\theta\,\boldsymbol{\sigma}_1 + r\sin\theta\,\boldsymbol{\sigma}_2}{r} = \cos\theta\,\boldsymbol{\sigma}_1 + \sin\theta\,\boldsymbol{\sigma}_2\,. \tag{14}$$

Now, we differentiate this vector:

$$\dot{\hat{\mathbf{r}}} = (-\sin\theta\,\boldsymbol{\sigma}_1 + \cos\theta\,\boldsymbol{\sigma}_2)\dot{\theta}\,. \tag{15}$$

On dividing this last equation through by $\dot{\theta}$ and using the calculus of differentials, we get

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \frac{d\hat{\mathbf{r}}/dt}{d\theta/dt} = -\sin\theta\,\boldsymbol{\sigma}_1 + \cos\theta\,\boldsymbol{\sigma}_2\,. \tag{16}$$

It's easy to show that $d\hat{\mathbf{r}}/d\theta$ is a unit vector and that $|\,d\hat{\mathbf{r}}/d\theta\,| = 1$.

Lemma:

$$\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{d\theta} = \boldsymbol{\sigma}_3\,, \tag{17}$$

where $\boldsymbol{\sigma}_3$ is the unit vector in the positive z direction.

Proof:

$$\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{d\theta} = \begin{bmatrix} \boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_2 & \boldsymbol{\sigma}_3 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} = \boldsymbol{\sigma}_3\,. \tag{18}$$

To proceed efficiently, we need to rewrite (3) as

$$\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}\,, \tag{19}$$

So, given that we can write $\mathbf{r} = r\hat{\mathbf{r}}$, then

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\,, \tag{20}$$

hence,

$$\begin{aligned} \mathbf{L} &= m\mathbf{r} \times \dot{\mathbf{r}} \\ &= m\mathbf{r} \times [\,\dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\,] \\ &= mr\,\mathbf{r} \times \dot{\hat{\mathbf{r}}} \\ &= mr^2\,\hat{\mathbf{r}} \times \dot{\hat{\mathbf{r}}} \\ &= mr^2\omega\,\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{d\theta} \\ &= mr^2\omega\,\boldsymbol{\sigma}_3\,. \end{aligned} \tag{21}$$

So, the magnitude of the angular momentum vector is given as

$$L = mr^2\omega\,. \tag{22}$$

Now, we're going to move slowly towards the equation of motion. To that end, let's get started with

$$\left\{ \begin{aligned} \hat{\mathbf{r}} &= \cos\theta\,\boldsymbol{\sigma}_1 + \sin\theta\,\boldsymbol{\sigma}_2\,, \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\,\boldsymbol{\sigma}_1 + \cos\theta\,\boldsymbol{\sigma}_2\,. \end{aligned} \right. \tag{23}$$

From this, note that $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$. On differentiating by time we get

$$\begin{cases} \dot{\hat{\mathbf{r}}} &= -\sin\theta\,\boldsymbol{\sigma}_1 + \cos\theta\,\boldsymbol{\sigma}_2 = \omega\,\hat{\boldsymbol{\theta}}\,, \\ \dot{\hat{\boldsymbol{\theta}}} &= -\cos\theta\,\boldsymbol{\sigma}_1 - \sin\theta\,\boldsymbol{\sigma}_2 = -\omega\,\hat{\mathbf{r}}\,. \end{cases} \tag{24}$$

On revisiting Eq. (20), we get

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\omega\,\hat{\boldsymbol{\theta}}\,. \tag{25}$$

Now, we need one more derivative, but I'll skip the in-between steps.

$$\ddot{\mathbf{r}} = (\dot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\omega})\,\hat{\boldsymbol{\theta}}\,. \tag{26}$$

Great, so now we have all these derivatives, but what to do with them? We must now connect them with a law of motion. We'll use the one provided by Newton. For starters, we have that Newton's Law of Gravity is a central force, thus,

$$\mathbf{F} = F\hat{\mathbf{r}} = m\ddot{\mathbf{r}}\,. \tag{27}$$

On combining these last two equations, we get that

$$\mathbf{F}/m = \ddot{\mathbf{r}} = (\dot{r} - r\dot{\theta}^2)\hat{\mathbf{r}}\,, \tag{28}$$

where we had to set the coefficient of the $\hat{\boldsymbol{\theta}}$ term in (26) to zero

$$2\dot{r}\dot{\theta} + r\ddot{\omega} = 0\,, \tag{29}$$

which provides us with a new constant of the planetary motion:

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0\,. \tag{30}$$

It's standard to use the symbol h in this equation, to get

$$h \equiv r^2\dot{\theta} = r^2\omega\,. \tag{31}$$

We can also relate h back to L:

$$h = L/m\,, \tag{32}$$

so that

$$r^2\omega = L/m\,. \tag{33}$$

3 Kepler's First Law and Newton's Second Law

Now we can rewrite (28) into a more convenient form:

$$\begin{aligned} \frac{F}{m} &= \ddot{r} - r\omega^2 \\ &= \ddot{r} - r\left(\frac{L}{mr^2}\right)^2 \\ &= \ddot{r} - \frac{L^2}{m^2r^3}\,. \end{aligned} \tag{34}$$

Once again, Newton's Law of Gravitation takes the form

$$F_g/m = -\frac{GM}{r^2}\,, \tag{35}$$

where M is the mass of the Sun and m is the mass of your planet of choice. Combining the last two equations, gives us

$$-\frac{GM}{r^2} = \ddot{r} - \frac{L^2}{m^2r^3}\,. \tag{36}$$

Solving for $\ddot{r}$ gives us

$$\ddot{r} = -\frac{GM}{r^2} + \frac{L^2}{m^2r^3}\,. \tag{37}$$

However, this equation is, in its present form, not so easy to solve. We get an easier equation by making the substitution $u = 1/r$, yielding,

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2}\,. \tag{38}$$

The solution to the homogeneous version of this equation is a linear combination of sines and cosines, but with a judicial choice of initial conditions, we can settle for just a cosine term (yes, this is legitimate). Thus,

$$u(\theta) = A\cos\theta + C\,. \tag{39}$$

Apparently, we should set $C = GMm^2/L^2$ and replace u by r:

$$\frac{1}{r(\theta)} = A\cos\theta + \frac{GMm^2}{L^2}\,. \tag{40}$$

Solving this for $r(\theta)$, we get

$$r(\theta) = \frac{1}{A\cos\theta + \dfrac{GMm^2}{L^2}}\,. \tag{41}$$

This is beginning to look like an equation for an ellipse in polar coordinates, isn't it? Recall that the standard polar form for an ellipse is given by

$$r = \frac{ed}{1 + e\cos\theta}\,. \tag{42}$$

So, to bring (41) into standard form, we can divide both numerator and denominator of the fraction in (41) by $\dfrac{GMm^2}{L^2}$ to get

$$r(\theta) = \frac{\dfrac{L^2}{GMm^2}}{\dfrac{AL^2}{GMm^2}\cos\theta + 1}\,. \tag{43}$$

If we now set

$$e \equiv \frac{AL^2}{GMm^2} \,. \tag{44}$$

Then we finally get

$$r(\theta) = \frac{ed}{1 + e\cos\theta} \,, \tag{45}$$

where $d = 1/A$.

So, what have we just accomplished? We have just proved by use of Newton's laws of motion and his universal law of gravitation that the planets, to first order approximation, orbit the Sun in ellipses, with the Sun at one of the foci. And that is Kepler's First Law of planetary motion.

4 Kepler's Second Law

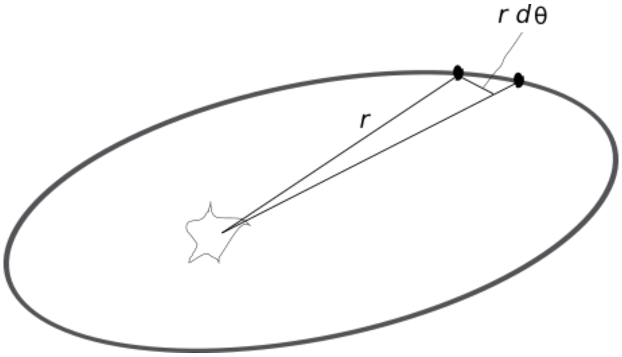

Figure 2. The planet's path is an ellipse with the Sun at one foci. The two dots represent the positions of the planet at two nearby points on the ellipse in time. The area swept out by the radial segment in an infinitesimal time is approximated by $\frac{1}{2}r^2 d\theta$.

The planet's path is an ellipse with the Sun at one foci. The two dots represent the positions of the planet at two nearby points on the ellipse in time. The area swept out by the radial segment in an infinitesimal time is approximated by $d\mathcal{A} = \frac{1}{2}r^2 d\theta$. So,

$$\frac{d\mathcal{A}}{dt} = \tfrac{1}{2}r^2\dot{\theta} = \tfrac{1}{2}r^2\omega = \frac{L}{2m} = \text{const}\,. \tag{46}$$

Hence, we're justified in claiming that

$$\Delta\mathcal{A} = \frac{L}{2m}\Delta t \,, \tag{47}$$

which is the essence of the claim in Kepler's Second Law of Motion. By integrating (46) by t over one complete period, we can further deduce that

$$\mathcal{A} = \frac{L}{2m}T \,, \tag{48}$$

where $\mathcal{A}$ is the area of the ellipse and T is the period of revolution of the planet about the Sun. Noting that the area of an ellipse is πab, and solving for T, we get

$$T = \frac{2m\pi ab}{L}\,, \tag{49}$$

where a and b are, respectively, the lengths of the semimajor and semiminor axes.

5 Kepler's Third Law

To prove Kepler's Third Law, we must show that

$$T^2 \propto a^3\,. \tag{50}$$

From (40), we have that

$$\frac{1}{r} = A\cos\theta + \frac{GMm^2}{L^2}\,. \tag{51}$$

And from the basic equation for an ellipse in polar coordinates, we have that

$$\frac{a(1-e^2)}{r} = 1 + e\cos\theta\,. \tag{52}$$

On multiplying (51) through by e/A, we get

$$\frac{L^2}{GMm^2}\frac{1}{r} = 1 + e\cos\theta\,, \tag{53}$$

where we used that

$$\frac{e}{A} = \frac{L^2}{GMm^2}\,. \tag{54}$$

Comparing (52) with (53), we get that

$$a(1-e^2) = \frac{L^2}{GMm^2}\,. \tag{55}$$

But we know from an earlier paper (Part 3: Directrix) that $b^2 = a^2(1-e^2)$, hence,

$$b^2 = \frac{L^2 a}{GMm^2}\,, \tag{56}$$

Therefore, from (49), and squaring:

$$\begin{aligned} T^2 &= \frac{(2m\pi ab)^2}{L^2} \\ &= \frac{(2m\pi ab)^2}{GMm^2b^2/a} \\ &= \frac{4\pi^2a^3}{GM}\,. \end{aligned} \tag{57}$$

Hence, we've shown that

$$T^2 \propto a^3\,. \tag{58}$$

6 Conclusion

When I took my mechanics class in physics as a young man all those decades ago, and saw the proofs of Kepler's Three Laws of Planetary Motion based on Newton's Laws of motion, I was not all that impressed. After all, it was not relativity. It was not quantum physics. But looking at it now without being unduly influenced by those short-sighted prejudices, I can see just how amazing this feat was.

Newton invented what it means to be a modern theoretical physicist. For the next two centuries physicists would struggle to complete the disciplines of thermodynamics and electrodynamics, and when they had, it was founded the same way as was Newton's mechanics: a bold set of postulates by which one could deduce extraordinary theorems to be tested against experiment or observation.

Of course, no flat-earther will be able to understand Newton's celestial mechanics, nor appreciate the profundity and beauty of his theoretical accomplishments.