Angle Bisectors of a Triangle Are Concurrent

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1 The Problem

Our problem is to show that the angle bisectors of a triangle are concurrent. Referencing Figure 1, we have that the angle bisectors from vertices **A** and **B** meet at point **D**. This proof will use trigonometry.

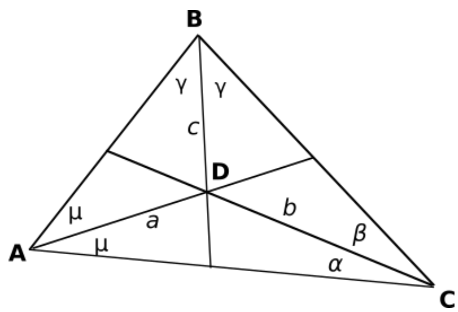

Figure 1. The angle bisectors from **A** and **B** meet at **D**. A line is extended from **C** though **D** to terminate on line segment$\overline{\mathbf{AB}}$. Letters a, b, c denote the lengths of line segments from the vertices to the point **D**.

2 The Solution

Our task at this point is to show that $\alpha = \beta$. The solution is simple: It follows from the repeated use of the Law of Sines:

$$\frac{\sin\mu}{b} = \frac{\sin\alpha}{a}\,, \tag{1a}$$

$$\frac{\sin\gamma}{a} = \frac{\sin\mu}{c}\,, \tag{1b}$$

$$\frac{\sin\beta}{c} = \frac{\sin\gamma}{b}\,. \tag{1c}$$

Now, just multiply them altogether:

$$\frac{\sin\mu}{b}\frac{\sin\gamma}{a}\frac{\sin\beta}{c} = \frac{\sin\alpha}{a}\frac{\sin\mu}{c}\frac{\sin\gamma}{b}\,, \tag{2}$$

which, after cancellations, yields

$$\sin\alpha = \sin\beta\,. \tag{3}$$

From this we get that

$$\alpha = \beta\,. \tag{4}$$

Done.