A Circle from Three Distinct Points — the Algebraic Proof

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1 Theorem

It's a well-known theorem that any three noncollinear, distinct points in a plane determine a circle that contains the three points. We will prove this theorem algebraically by finding the coordinates of the center of this circle as functions of the coordinates of the three points.

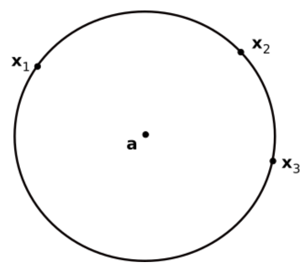

Figure 1. The three distinct, noncollinear points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ of a given plane are shown as lying on a circle of center point $\mathbf{a}$. The proof of the fact that these points do lie on a circle will be demonstrated by solving for the coor- -dinates of the center point $\mathbf{a}$ as functions of the three given points.

2 Proof

Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ be three distinct, noncollinear points in a plane. Further, let $\mathbf{a}$ be the presumptive center point of a circle that contains the three given points. Let the coordinates of the ith given point be (x_i, y_i). If these three points really do lie on a circle of center point $\mathbf{a} = (a_1, a_2)$, then they must all satisfy the same distance equation of some as yet undetermined radius r:

$$(x_1 - a_1)^2 + (y_1 - a_2)^2 = r^2 \,, \tag{1}$$

$$(x_2 - a_1)^2 + (y_2 - a_2)^2 = r^2 \,, \tag{2}$$

$$(x_3 - a_1)^2 + (y_3 - a_2)^2 = r^2 \,. \tag{3}$$

On expanding the first two of these equations and simplifying, we get

$$x_1^2 - 2x_1a_1 + a_1^2 + y_1^2 - 2y_1a_2 + a_2^2 = r^2\,, \tag{4}$$

$$x_2^2 - 2x_2a_1 + a_1^2 + y_2^2 - 2y_2a_2 + a_2^2 = r^2\,. \tag{5}$$

Next, we eliminate r between these last two equations, to get

$$x_1^2 - 2x_1a_1 + +y_1^2 - 2y_1a_2 = x_2^2 - 2x_2a_1 + y_2^2 - 2y_2a_2\,. \tag{6}$$

This equation can be rewritten as

$$(x_2 - x_1)a_1 + (y_2 - y_1)a_2 = \lambda\,, \tag{7}$$

where

$$\lambda \equiv \tfrac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2)\,. \tag{8}$$

I introduced the number λ just to simplify the equation (7). Note that it is just a number because the values it's dependent on, x_1, x_2, y_1, and y_2 , are given to us. Now we need to include Eq. (3) into the mix. To that end, we permute the indicies on the x's and y's in (6), (7), and (8) by $2 \to 1 \to 3$, to get:

$$x_3^2 - 2x_3a_1 + y_3^2 - 2y_3a_2 = x_1^2 - 2x_1a_1 + y_1^2 - 2y_1a_2\,. \tag{9}$$

This equation can be rewritten as

$$(x_1 - x_3)a_1 + (y_1 - y_3)a_2 = \mu\,, \tag{10}$$

where

$$\mu \equiv \tfrac{1}{2}(x_1^2 - x_3^2 + y_1^2 - y_3^2)\,. \tag{11}$$

Eqs. (7) and (10) can be put into the matrix form:

$$\begin{bmatrix}\lambda \\ \mu\end{bmatrix} = \begin{bmatrix}(x_2 - x_1) & (y_2 - y_1) \\ (x_1 - x_3) & (y_1 - y_3)\end{bmatrix} \begin{bmatrix}a_1 \\ a_2\end{bmatrix}\,. \tag{12}$$

Under the assumption that the determinant D of the 2×2 matrix of (12) is not zero, we can invert this equation to get

$$\begin{bmatrix}a_1 \\ a_2\end{bmatrix} = \begin{bmatrix}(x_2 - x_1) & (y_2 - y_1) \\ (x_1 - x_3) & (y_1 - y_3)\end{bmatrix}^{-1} \begin{bmatrix}\lambda \\ \mu\end{bmatrix} = \frac{1}{D}\begin{bmatrix}(y_1 - y_3) & -(y_2 - y_1) \\ -(x_1 - x_3) & (x_2 - x_1)\end{bmatrix} \begin{bmatrix}\lambda \\ \mu\end{bmatrix}\,.$$

Expanding this to better see a_1 and a_2, we have

$$a_1 = \frac{\lambda(y_1 - y_3) - \mu(y_2 - y_1)}{D}\,, \tag{13}$$

$$a_2 = \frac{-\lambda(x_1 - x_3) + \mu(x_2 - x_1)}{D}\,, \tag{14}$$

where

$$D = (x_2 - x_1)(y_1 - y_3) - (x_1 - x_3)(y_2 - y_1)\,. \tag{15}$$

And now that we know a_1 and a_2, we can solve for the circle's radius r by solving any one of (1), (2), or (3) for r.

3 Trial Problem

Let's test our formulas by choosing three 'easy' points to calculate with. Let

$$
\begin{aligned}
(x_1, y_1) &= (0, 1)\,,\\
(x_2, y_2) &= (1, 0)\,,\\
(x_3, y_3) &= (2, 1)\,,
\end{aligned}
$$

which I know to be three points on a circle of radius unity, centered on the point $(1, 1)$. So, let's prove these facts. First we calculate D, λ, and μ.

$$D = (x_2 - x_1)(y_1 - y_3) - (x_1 - x_3)(y_2 - y_1) = (1-0)(1-1) - (0-2)(0-1) = -2\,.$$

And

$$\lambda = \tfrac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2) = \tfrac{1}{2}(1^2 - 0^2 + 0^2 - 1^2) = 0\,.$$

Finally,

$$\mu = \tfrac{1}{2}(x_1^2 - x_3^2 + y_1^2 - y_3^2) = \tfrac{1}{2}(0^2 - 2^2 + 1^2 - 1^2) = -2\,.$$

Then

$$
\begin{aligned}
a_1 &= \frac{\lambda(y_1 - y_3) - \mu(y_2 - y_1)}{D} = \frac{0 - (-2)(0-1)}{-2} = 1\,,\\
a_2 &= \frac{-\lambda(x_1 - x_3) + \mu(x_2 - x_1)}{D} = \frac{0 + (-2)(1-0)}{-2} = 1\,.
\end{aligned}
$$

To calculate r, we choose Eq. (1)[1] and write

$$
\begin{aligned}
r &= \sqrt{(x_1 - a_1)^2 + (y_1 - a_2)^2}\\
&= \sqrt{(0-1)^2 + (1-1)^2}\\
&= 1\,,
\end{aligned}
$$

which is what we expected.

[1] I could have chosen either Eq. (2) or Eq. (3), as well, since all three equations should yield the same r.