

Two Distinct Non-Parallel Planes Intersect in a Line

P. Reany

May 4, 2022

1 Theorem

It's easy to visualize that two distinct non-parallel planes in \mathcal{R}^3 intersect in a line. We'll prove this by parametrizing the intersection of these planes by a single variable, proving that the intersection space is one-dimensional.

Let the two planes be given by the coordinate equations

$$a_1x + b_1y + c_1z = d_1, \quad (1)$$

$$a_2x + b_2y + c_2z = d_2, \quad (2)$$

2 Proof

We'll choose x as our independent variable, and solve for y and z in terms of x . To accomplish that, we'll rewrite our original pair of equations as

$$b_1y + c_1z = d_1 - a_1x, \quad (3)$$

$$b_2y + c_2z = d_2 - a_2x. \quad (4)$$

Now we rewrite these in matrix form

$$\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} d_1 - a_1x \\ d_2 - a_2x \end{bmatrix}. \quad (5)$$

Expanding this with Cramer's Rule, we have

$$y = \frac{\begin{vmatrix} d_1 - a_1x & c_1 \\ d_2 - a_2x & c_2 \end{vmatrix}}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}, \quad (6)$$

$$z = \frac{\begin{vmatrix} b_1 & d_1 - a_1x \\ b_2 & d_2 - a_2x \end{vmatrix}}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}. \quad (7)$$