Distance between Centers of Two Circles

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Abstract

I found this little gem of a fun problem to solve in a website for unwanted 'helpful' math problems for high school students. See if you enjoy solving it. But first, you have to decide the method of approach to the solution.

Introduction:
This problem is found at webpage:

`https://delong.typepad.com/sdj/2005/07/100_interesting.htm`

and is titled '100 Interesting Mathematical Problems, Exercises, Puzzles, and Diversions'.

The problem is:

> In a 5 by 12 rectangle, one of the diagonals is drawn and circles are inscribed in both right triangles thus formed. Find the distance between the centers of the two circles.

Step One: The initial setup

I can just imagine the complaint of those students who hate word problems to go something like, "I have no idea where to even begin!" My answer: How about we start with a figure?

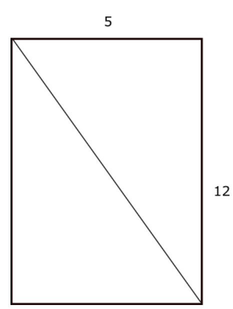

Figure 1. A 5 × 12 rectangle with one diagonal already drawn.

My solution begins by asking myself, Which geometry do I want to try to find a solution within? There's always Geometric Algebra, of course. Or even Gibbs's vector algebra. No, not this time. But what about classical Euclidean Geometry? No, not for me, anyway. What's left, then? Yes, of course: Analytic Geometry! For if we know the coordinates of the two centers of the circles, we can just apply the distance formula to get the distance between them.

The distance d between two points (x_1, y_1) and (x_2, y_2) is given as

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \tag{1}$$

Okay, then, analytic geometry it is! But where do we place the origin? It can be placed anywhere, but we ought to place it so as to simplify the computations. My choice for the origin, then, is revealed in Fig. 2.

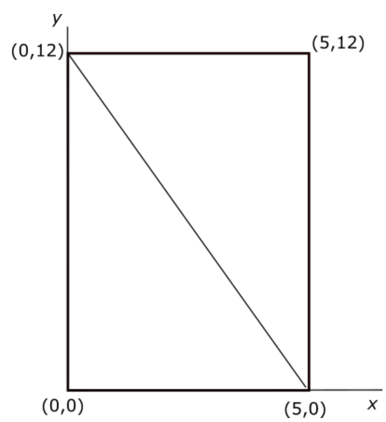

Figure 2. The rectangle has been coordinatized in the x, y-plane.

Obviously I already drew-in one of the diagonals. Either one will do: Either choose the diagonal from $(0, 12)$ to $(5, 0)$ or draw the diagonal from $(0, 0)$ to $(5, 12)$.[1] The reason we can choose either diagonal is because of the huge amount of symmetry a rectangle affords us. After all, the interior angles are all right angles, and the opposite sides are parallel to each other and of the same lengths in pairs.

Step Two: Place and Coordinatize the Circles

Our next task is to inscribe circles in both the bottom left and top right portions of the rectangle (effectively, two triangles), and then label the relevant points. See Fig. 3.

[1] By the way, we could have placed the long side of the rectangle on the x-axis instead of on the y-axis.

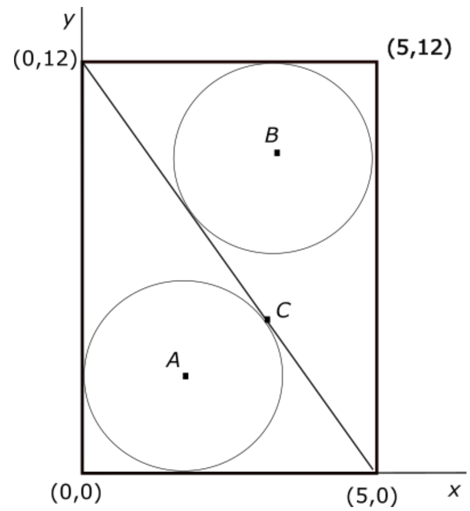

Figure 3. Two circles have been inscribed in the rectangle as per the instructions. Points A and B label the two centers of the circles. And they are the incenters of their respective triangles.

According to Euclidean geometry, the *incenter*[2] of a triangle is the center of the circle inscribed in the triangle. I define the radii of the two circles as r. See Fig. 4 for added details.

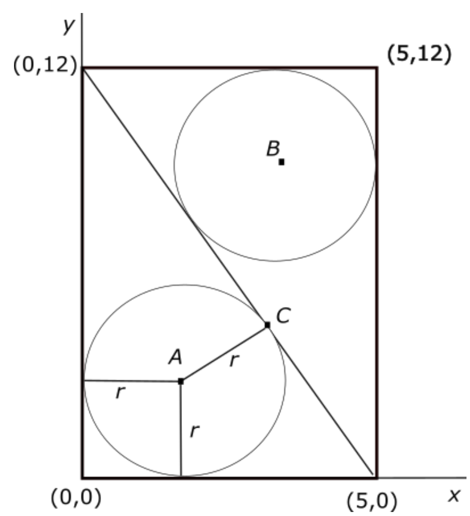

Figure 4. Each line segment from the center of the circle to any point on the circle is perpendicular to the tangent line at that point. Thus, we immediately know the coordinates of point A.

[2]According to Wikipedia, the *incenter* of a triangle is the point that is equidistant from all three sides of the triangle.

The meaning of the inscribed circle is that it touches each of the three sides so that each side is tangent to the circle at the points of tangency. Hence, each line segments going from the center of the circle to a point on the circle is perpendicular to the tangent line at that point.

We now have enough information to coordinatize point B, and we rely heavily on the congruency of the two triangles. See Figure 5.

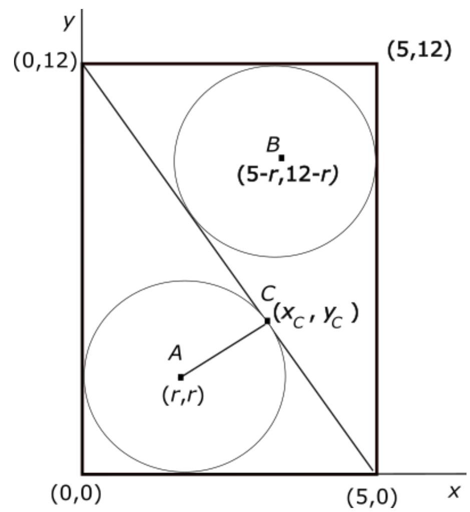

Figure 5. Because of the congruency of the two triangles, point B stands in relation to point $(5, 12)$ as point A stands in relation to point $(0, 0)$.

Step Three: Write Down the Simultaneous Equations and Solve Them

Even though we haven't been asked to solve for the coordinates of point C, they are pivitol to finding the value of r (at least within this present method of proof), and by which we can determine the coordinates of points A and B.

Now, point C lies on three 'objects', which will provide for us three equations to be simultaneously solved. That is, point C is a solution of the diagonal-line equation

$$y = -\frac{12}{5}x + 12\,. \tag{2a}$$

And it's also a solution of the line perpendicular to it:[3]

$$\frac{y-r}{x-r} = -\big[-\frac{5}{12}\big] = \frac{5}{12}\,. \tag{2b}$$

Lastly, it's also a solution for the equation of the circle centered at $A = (r, r)$:

$$r^2 = (x-r)^2 + (y-r)^2\,. \tag{2c}$$

[3]Here we have used the fact that for two mutually perpendicular lines in the plane, the product of their slopes (m_1 for one line and m_2 for the other) is given by $m_1 m_2 = -1$.

So, using the coordinates of point C, namely (x_C, y_C), we have

$$y_C = -\frac{12}{5}x_C + 12\,. \tag{3a}$$

And the line perpendicular to it:

$$\frac{y_C - r}{x_C - r} = \frac{5}{12}\,. \tag{3b}$$

And the equation for the circle centered at A:

$$r^2 = (x_C - r)^2 + (y_C - r)^2\,. \tag{3c}$$

On solving these last three equations simultaneously for r (I used Wolfram-Alpha to do this), we get $r = 2$; hence, substituting this into the distance formula (1), we get for the distance between the circle centers

$$\begin{aligned} d &= \sqrt{[(5-r)-r]^2 + [(12-r)-r]^2} \\ &= \sqrt{[3-2]^2 + [10-2]^2} \\ &= \sqrt{1+64} \\ &= \sqrt{65}\,. \end{aligned} \tag{4}$$

The command I used in WolframAlpha was

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solve [y=-\frac{12}{5}x+12,\frac{y-r}{x-r}=\frac{5}{12},r^2=(x-r)^2+(y-r)^2]
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By the way, the on-line 'tutor' who calls himself *richard1234*, arrived at the same solution by a different method. His solution is found at:
https://www.algebra.com/algebra/homework/word/geometry/Geometry_Word_Problems.faq.question.574556.html

Conclusion

This is the kind of problem that one should try to solve from a number of different perspectives.