# Cross Ratio 1: The Geometry of Areas

P. Reany

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**Abstract**

This paper proves the invariance of the cross ratio using the areas of the triangles in the 'cross ratio' figure.

## 1 Introduction

We will prove that a certain quantity know as the 'cross ratio' of four numbers in a projective planar figure is an invariant under central projection.

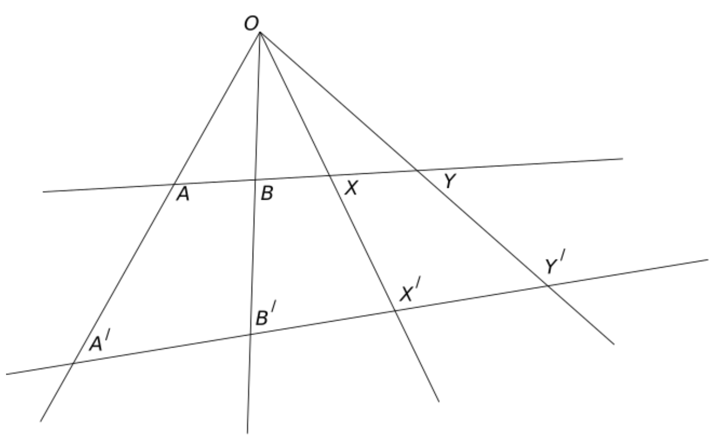


Figure 1. Out of the pencil of lines going through point $O$ in the plane, we choose four of them. Then we choose two distinct lines in the plane that do not intersect point $O$. On one of these transverse lines that intersect the four pencils, we denote the four points of intersections as $A$, $B$, $X$, $Y$. Similarly, on the second line, we label the corresponding points as $A'$, $B'$, $X'$, $Y'$.

Now, we shall prove that the invariant 'cross ratio' of four points is given as

$$\frac{XB}{XA}\frac{YA}{YB} = \frac{X'B'}{X'A'}\frac{Y'A'}{Y'B'}\,. \tag{1}$$

There are many proofs to this theorem. I recently saw one of these proofs that only used the Law of Sines, repeatedly and some algebra, and that's perfectly fine. I used the method of the areas of the triangles and that worked for me. Sometimes you just have to use the method that you're most comfortable with.[1]

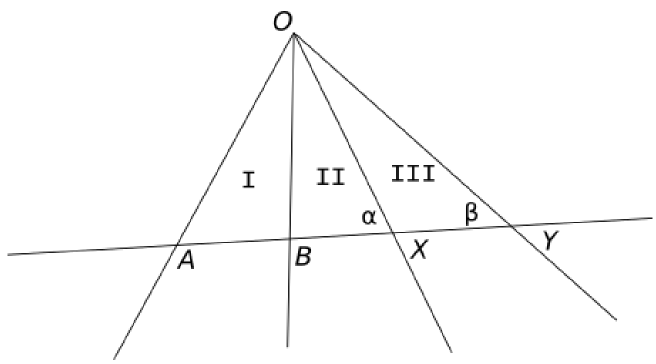


Figure 2. The roman numeral letters are denoting the areas of the three triangles that they are embedded within. The greek letters $\alpha$ and $\beta$ refer to angles at points $X$ and $Y$, respectively. I'll make use of the fact that triangles have areas that can be calculated in two different ways. That sounds like an equation afoot.

## 2 Proof:

The situation is described in Fig. 2. I intend to overload the roman numerals to use them, not only to represent regions, but also to represent the angles between the rays that radiate from point $O$.

One way to find twice the area of a triangle is to take the product of any two sides and then multiply that by the sine of the angle between those sides. Thus,

$$\begin{cases} 2\text{Area}_{\mathrm{II}} & = OX \cdot XB \sin\alpha = OX \cdot OB \sin\angle\mathrm{II}\,, \\ 2\text{Area}_{\mathrm{I}+\mathrm{II}} & = OX \cdot XA \sin\alpha = OX \cdot OA \sin\angle(\mathrm{I}+\mathrm{II})\,. \end{cases} \tag{2}$$

And then we have

$$\begin{cases} 2\text{Area}_{\mathrm{II}+\mathrm{III}} & = OY \cdot YB \sin\beta = OY \cdot OB \sin\angle(\mathrm{II}+\mathrm{III})\,, \\ 2\text{Area}_{\mathrm{I}+\mathrm{II}+\mathrm{III}} & = OY \cdot YA \sin\beta = OY \cdot OA \sin\angle(\mathrm{I}+\mathrm{II}+\mathrm{III})\,. \end{cases} \tag{3}$$

It's clear that if we divide one of the equations of Case Eq. (2) by the other, that we'll get some cancellation, yielding:

$$\frac{XB}{XA} = \frac{OB \sin\angle\mathrm{II}}{OA \sin\angle(\mathrm{I}+\mathrm{II})}\,. \tag{4}$$

[1] Obsessively making a proof shorter for its own sake, may render the proof less clear.

Comparing this last result with (1), it appears that we're on the right track. So, doing similarly to Case Eq. (3), we get

$$\frac{YB}{YA} = \frac{OB \sin\angle(\mathrm{II}+\mathrm{III})}{OA \sin\angle(\mathrm{I}+\mathrm{II}+\mathrm{III})}\,. \tag{5}$$

Now, we divide (4) by (5) to get

$$\frac{XB}{XA}\frac{YA}{YB} = \frac{\sin\angle\mathrm{II}}{\sin\angle(\mathrm{I}+\mathrm{II})}\frac{\sin\angle(\mathrm{I}+\mathrm{II}+\mathrm{III})}{\sin\angle(\mathrm{II}+\mathrm{III})}\,. \tag{6}$$

It's quite obvious that if we had applied the same procedure to the line containing the primed points (see Fig. 1), we would have ended up with

$$\frac{X'B'}{X'A'}\frac{Y'A'}{Y'B'} = \frac{\sin\angle\mathrm{II}}{\sin\angle(\mathrm{I}+\mathrm{II})}\frac{\sin\angle(\mathrm{I}+\mathrm{II}+\mathrm{III})}{\sin\angle(\mathrm{II}+\mathrm{III})}\,. \tag{7}$$

Therefore, by the transitive property of equality over the reals, we get that

$$\frac{XB}{XA}\frac{YA}{YB} = \frac{X'B'}{X'A'}\frac{Y'A'}{Y'B'}\,. \tag{8}$$