Projective Geometry 4: Desargues's Theorem with Coordinate-Free Proof

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Abstract

This paper proves Desargues's Theorem using a coordinate-free proof.

1 Introduction

We prove Desargues's Theorem using the coordinate-free form of Gibbs's vector algebra. The proof proceeds in two steps. First, we find an expression that would prove collinearity if a certain condition holds, and then we use the various constraints to find the missing pieces to the proof.

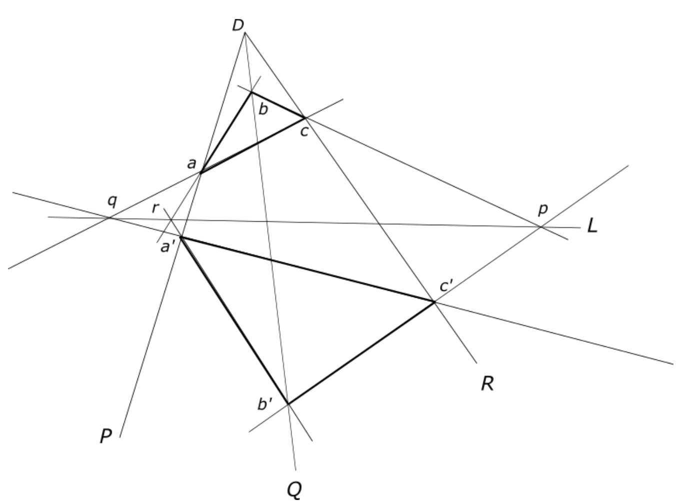

Figure 1. Desargues's Theorem. Triangles abc and $a'b'c'$ are arranged in the plane such that lines P, Q, and R meet at point D.

Desargues's Theorem: If two triangles in a given plane are perspective from a point, they are also perspective from a line. The easiest way to understand this claim is from the diagram in Figure 1. Triangles abc and $a'b'c'$ are arranged in the plane such that lines P, Q, and R meet at point D, making the two triangles "perspective from a point." When the corresponding sides of the triangles are extended, they meet in pairs in three points which lie on a line. In Figure 1, this line of collinearity is L.

We start with the concurrence constraint at point D, explicitly,

$$D = [aa'bb'] = [bb'cc'] = [aa'cc'] \,. \tag{1}$$

I'll expand this 'one' equation to three:

$$[aa'bb'] = [bb'cc'] \,, \tag{2a}$$
$$[aa'bb'] = [aa'cc'] \,, \tag{2b}$$
$$[bb'cc'] = [aa'cc'] \,. \tag{2c}$$

I took care to make sure, according to the right-hand rule for cross products, that all the point locations[1] in (1) came out positive.

Now, given the above constructions, we are to show that the triple scalar product of p, q, r is zero[2]:

$$[pqr] = 0 \,, \tag{3}$$

thus ensuring that points p, q, r are collinear.

2 The Proof

The point locations for p, q, and r can be expressed as

$$p = [cbc'b'] \,, \tag{4a}$$
$$q = [cac'a'] \,, \tag{4b}$$
$$r = [baa'b'] \,. \tag{4c}$$

I have not ensured that the point locations for these points are positive because the only use I will make of them is in the triple scalar product (3), and the over-all sign of $[pqr]$ is irrelevant for this proof.

All right, time for calculations. But before we get to $[pqr]$, let's get an expression for $[qr]$ first, where

$$\begin{aligned}[qr] &= q \times r \\ &= [cac'a'] \times [baa'b'] \,.\end{aligned} \tag{5}$$

There are many ways to expand these point locations using the formulas

$$[xyzw] = [y][xzw] - [x][yzw] \quad \text{or} \tag{6a}$$
$$[xyzw] = [z][xyw] - [w][xyz] \,. \tag{6b}$$

[1]A *point location* is the meet of two lines in the projective plane, and it is represented by a triple cross product, using four distinct points of the plane.

[2]Remember that $[pqr] \equiv p \cdot [qr] = p \cdot (q \times r)$.

I chose to expand $[qr]$ as follows

$$\begin{aligned}[qr] &= ([a][cc'a'] - [c][ac'a']) \times ([a][ba'b'] - [b][aa'b']) \\ &= -[ba][cc'a'][aa'b'] - [ca][ac'a'][ba'b'] + [cb][ac'a'][aa'b']\,. \end{aligned} \tag{7}$$

I chose to expand $[p]$ as

$$[p] = [b][cc'b'] - [c][bc'b']\,. \tag{8}$$

Thus, we get for $[pqr]$, after some simplifications

$$\begin{aligned}[pqr] &= -[bca][cc'b'][ac'a'][ba'b'] + [cab][bc'b'][cc'a'][aa'b'] \\ &= -[abc](-[cb'c'])(-[aa'c'])[ba'b'] + [abc](-[bb'c'])(-[ca'c'])[aa'b'] \\ &= -[abc]\big([cb'c'][aa'c'][ba'b'] - [bb'c'][ca'c'][aa'b']\big)\,. \end{aligned} \tag{9}$$

And this is essentially as far as we can go in Step One of the proof. Now we look for some way to pull out some other factor or factors from the RHS of (9).

STEP TWO:

There are so many ways to seek these constraints out that one can easily be overwhelmed by the effort. Fortunately, I can give the end result of my search.

I expanded the point locations in (2a), using (6b) on the LHS and (6a) on the RHS, to get

$$[b][aa'b'] - [b'][aa'b] = [b'][bcc'] - [b][b'cc']\,. \tag{10}$$

On dotting this through by the vector $[a'b']$ and simplifying, we get

$$[aa'b'] = [cb'c']\,. \tag{11}$$

Using this last result in (9), we have

$$[pqr] = -[abc][aa'b']\big([aa'c'][ba'b'] - [bb'c'][ca'c']\big)\,. \tag{12}$$

Now to establish (3), all we have to do is to show that

$$[aa'c'][ba'b'] - [bb'c'][ca'c'] = 0\,. \tag{13}$$

To accomplish this last task, we expand (2a) as follows

$$[a'][abb'] - [a][a'bb'] = [c][bb'c'] - [c'][bb'c]\,. \tag{14}$$

On dotting this through by the vector $[a'c']$ and simplifying, we get

$$[aa'c'][ba'b'] = [bb'c'][ca'c']\,, \tag{15}$$

which is equivalent to (13). And thus we have shown what was required.