# Projective Geometry 5: Desargues's Theorem with Analytic Proof Adapted from Dorwart

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**Abstract**

This paper proves Desargues's Theorem using an adaptation of Dorwart's analytic proof from his book.

## 1 Introduction

We prove Desargues's Theorem using a version of Harold L. Dorwart's proof from his book, *The Geometry of Incidence*, Prentice-Hall, 1966, pp. 95–98.[1]

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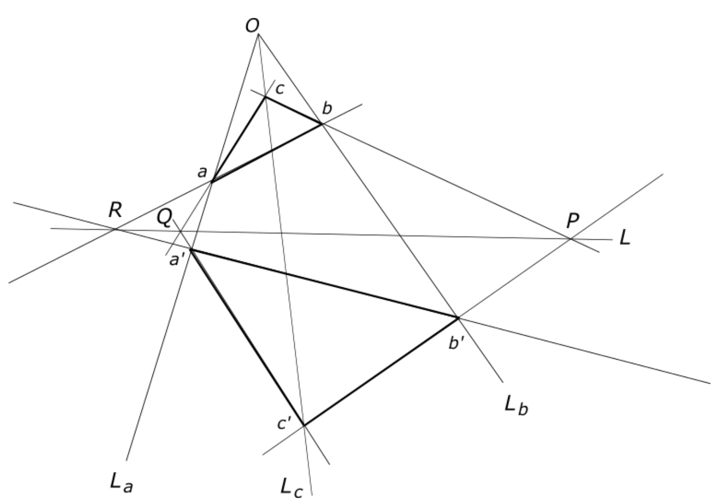


Figure 1. Desargues's Configuration. This figure is an adaptation of the figure I used in the last paper that used coordinate-free vector algebra to prove the theorem.

[1]The necessary preparation to understand the theory and notations used in this paper is provided in the first paper of this series on projective geometry.

Desargues's Theorem: If two triangles in a given plane are perspective from a point, they are also perspective from a line.[2]

The easiest way to understand this claim is from the diagram in Figure 1. Triangles $abc$ and $a'b'c'$ are arranged in the plane such that lines $L_a$, $L_b$, and $L_c$ meet at point $O$, making the two triangles "perspective from a point." When the corresponding sides of the triangles are extended, they meet in pairs, forming three points which lie on a line, which is depicted as $L$.

## 2 Proof:

There are three basic steps to Dorwart's proof:

STEP 1: Assign the four canonical points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$ to four distinct points on the figure, no three of which are collinear. These four points can be thought of as seeding the figure (See Figure 2.).

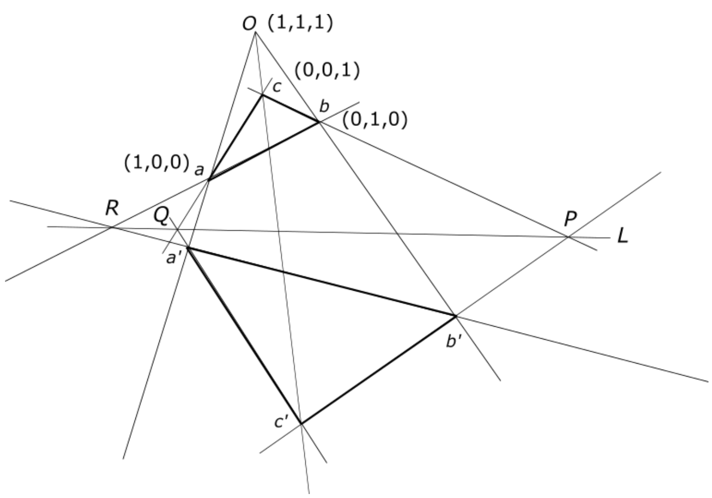


Figure 2. This figure shows the initial four point coordinates as given by Dorwart on pages 96–97. However, no three of those points may be collinear.

STEP 2: Calculate appropriate coordinates (with suitable restrictions) to the rest of the points.

STEP 3: Verify that the triple scalar product[3] of meet points $P$, $Q$, and $R$ is zero, confirming that the three points are collinear. Note: It would be messy to try to calculate the meet points of $P$, $Q$, and $R$ by use of the triple scalar

[2]There is also a version of Desargues's Theorem for 3-space, which is not covered here.

[3]The 'triple scalar product' of three arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is given by $[\mathbf{abc}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

product because we would have to introduce new parameters for each point. So, instead, we just calculate their coordinates by using their 'projective locations'.[4]

Now, I'll assign coordinates to the rest of the points of the figure in keeping with Dorwart's 'analytic' proof.[5]

Having seeded the figure with the four canonical points, we can determine suitable coordinates for the rest of the points, beginning with $a'$. Since $a'$ is collinear with points $O$ and $a$, it is to be solved for by employing the following constraint[6]:

$$[a'Oa] = 0\,. \tag{1a}$$

Next we assign variable coordinates to $a'$

$$a' = (r, s, t)\,. \tag{1b}$$

Then (1a) becomes

$$\begin{vmatrix} r & s & t \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0\,, \tag{1c}$$

yielding

$$s - t = 0\,. \tag{2}$$

So, we choose $s = t = 1$ for simplicity, and then $r$ is an arbitrary real number, though not equal to unity, to keep it distinct from point $O$. So,

$$a' = (r, 1, 1)\,, \tag{3}$$

where $r \neq 1$

Likewise, for $b'$, we have that

$$[b'Ob] = 0\,. \tag{4a}$$

As the we did in the last case, we assign variable coordinates to $b'$

$$b' = (u, v, w)\,, \tag{4b}$$

and (4a) becomes

$$\begin{vmatrix} u & v & w \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 0\,, \tag{4c}$$

[4]The 'point location' of a point $M$ in the projective plane is calculated by $M = [abcd]$, where $M$ is the meet of the two lines defined by $[ab]$ and $[cd]$.

[5]By 'analytic', I take it to mean 'any proof that relies on coordinates', which, so far as I can discern, began with the invention of homogeneous-coordinate model of the projective plane, the term being a direct generalization of Descartes's analytic (coordinate) geometry. But make no mistake, homogeneous coordinates for the projective plane means no other than a coordinate-laden version of Gibbs's vector algebra.

[6]The symbol $[a'Oa]$, without commas in it, is not to be confused with so-called 'line coordinates', such as $[\alpha, \beta, \gamma]$, which is not used in this paper, though it is used by Dorwart in his proof.

yielding

$$w - u = 0\,. \tag{5}$$

And as before, we choose $w = u = 1$ for simplicity, leaving $v$ is an arbitrary real number, other than unity. So

$$b' = (1, v, 1)\,, \tag{6}$$

where $v \neq 1$. By similar reasoning, we get $c' = (1, 1, h)$, where $h \neq 1$. This brings us to the updated figure:

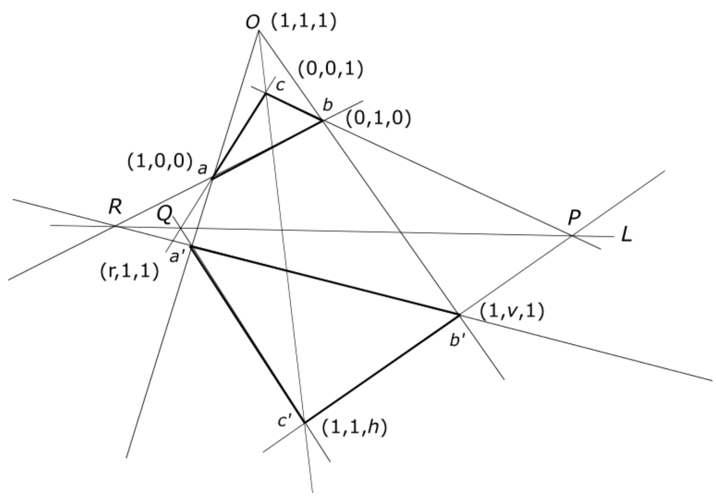


Figure 3. This figure shows the coordinates of the additional three point we have just solved for.

We now add the constraint that $a'$, $b'$, and $c'$ form a nondegenerate triangle, or that $[a'b'c'] \neq 0$. This results in the relation[7]

$$[a'b'c'] = rvh - r - v - h + 2 \neq 0\,. \tag{7}$$

Next we calculate the coordinates of points $P$, $Q$, and $R$ (each by their projective locations), so we can calculate their triple scalar product.

$$\begin{aligned}[P] &= [bcb'c'] \\ &= [c][bb'c'] - [b][cb'c'] \\ &= (0, v-1, 1-h)\,,\end{aligned} \tag{8a}$$

[7]Dorwart does not explain the importance of this constraint in practice. Furthermore, the Gibbs's vector method I used in the previous paper does not require any such similar constraint.

and

$$
\begin{aligned}
[Q] &= [a'c'ac] \\
&= [c'][a'ac] - [a'][c'ac] \\
&= (r-1, 0, 1-h)\,,
\end{aligned}
\tag{8b}
$$

and

$$
\begin{aligned}
[R] &= [a'b'ab] \\
&= [b'][a'ab] - [a][b'ab] \\
&= (1-r, v-1, 0)\,.
\end{aligned}
\tag{8c}
$$

Finally, we can now calculate $[PQR]$. However, I leave it to the reader to show that $[PQR] = 0$, establishing that points $P$, $Q$, and $R$ are collinear. (Hint: Just form a determinant out of the three points.)

# 3 Conclusion

After concluding this, the fifth paper in this series on projective geometry, I think I'm in a position to make a few comments of comparison on the methods of theorem proving between that of the coordinate-free Gibbs's approach vs. the homogeneous-coordinate (analytic) approach provided by Dorwart in his proofs. And, to that end, I ask the reader's indulgence as I present a rather long exposè at this point of the series.

First off, I think the three most important take-away points to note at this moment are

1. That the homogeneous-coordinate (analytic) approach is just Gibbs's vector algebra, obscured by lack of direct reference to it, and by its use of extra terminology, like 'line coordinates'.

2. That Gibbs's vector algebra could be repurposed from its typical modern-day metrical uses in Euclidean geometry and in physics, to use at all in projective geometry is beyond being merely unintuitive, encroaching on the mystical.

3. That among the abundant literature available on-line on projective geometry of the real plane, virtually none of it makes mention of the fact that it's really just a version of Gibbs's vector algebra, though there is an occasional reference to it being under the auspices of linear algebra, instead.

Now that I've had a chance to look at the life of a projective geometer from both sides now[8], I should like to compare and contrast their relative strengths and weaknesses.[9]

[8]Yes, this is a direct reference to Joni Mitchell's song by that name.

[9]Later in this series, I intend to comment on how Geometric (Clifford) algebra fares on formulating projective geometry.

So far in this series, the problems (theorems) have come in two forms: Given a particular configuration of points and lines in a plane, A) determine if three points are collinear, or B) determine if three lines are concurrent. Viewed from the highest level, both approaches to theorem proving are the same:

1. State the given information in vector form.
2. State the ShowThat condition in vector form.
3. Find a suitable logical connection between points 1. and 2. above.

And it's in this last item that the two approaches diverge so much from each other.

In the approach demonstrated by Dorwart, one assigns the first four canonical points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$ to four distinct points on the figure, no three of which are collinear. These four points can be thought of as seeding the figure. But Dorwart himself leaves the justification for this seeding as a matter of faith for the reader. Furthermore, he never explained his rationale for why he chose the points he chose as the proper placement of the canonical points on the figure. Surely, they weren't dispersed randomly.

On pages 82–83 of the text, on the verge of Dorwart proving the first significant theorem of the book (i.e., Pappus's Theorem), he says this:

> Before starting the proof of this theorem, we remind the reader of the so-called "analytic proofs" of theorems concerning triangles, etc., that are given in all textbooks on analytic geometry. The simplicity of these proofs arises from a proper location of the figure relative to the coordinate axes. If one wishes to prove a theorem about *any* triangle, he may take one vertex at the origin, another vertex on say the $x$-axis, and the third at some general point not on either axis. . .The zeros that appear in the coordinates materially reduce the labor in later computations. . .Similar simplifications are available for analytic proofs in the real projective plane. For example, it can be shown that for any four points in the plane, no three of which are collinear, may be taken as $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$.

Thus, Dorwart leaves us at this point with many mysteries and loose ends: For example, why can't we use the origin in this canonical set of points, making a total of five canonical points to distribute over the figure? Of course, we know why in the presentation of the homogeneous model of the real projective plane: because the origin is not in the projective plane because of those nasty ideal points at infinity. And Dorwart had to spend a lot of time to establish these ideal elements, making them seem, for historical reasons, to be central to the operation of projective geometry, yet the existence of points at infinity never came up in his proof. And we have no rationale why we can use the canonical four at all. All Dorwart says in their justification is that 'it can be shown that'.

And the last aspect of this homogeneous coordinate approach to proving theorems, requires the solver to introduce real variables as components of the points established after setting the canonical four have been chosen. What's worse, these variables may end up with mysterious constraints on them, as we saw in the equation

$$[a'b'c'] = rvh - r - v - h + 2 \neq 0\,, \tag{9}$$

which leaves me wondering how to apply this equation in real life. The coordinate-free approach does not require introducing these extra variables and their enigmatic constraints.

Let's take a look at a classic problem of Euclidean geometry. Inscribe a triangle into a circle such that the largest side of the triangle is the diameter of the circle. Show that the triangle is right. In other words, show that the inscribed triangle is a right triangle.

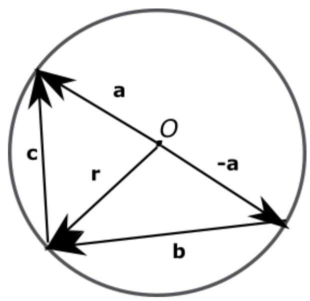


Figure 4. This figure shows a triangle inscribed in a circle, the sides of the triangle labeled with convenient free vectors. The point $O$ is, of course, the center of the circle.

One way to show that the triangle is a right triangle is to show that the dot product of **b** with **c** is zero. So, let's build some obvious equations from this diagram:

$$\mathbf{a} = \mathbf{r} + \mathbf{c}\,, \tag{10a}$$

$$\mathbf{r} = -\mathbf{a} + \mathbf{b}\,. \tag{10b}$$

And let's not forget to take into account the constraint that all vectors with base points at the center of the circle and tips on the circle have the same length:

$$\mathbf{a}^2 = \mathbf{r}^2\,. \tag{11}$$

From which we have that

$$\begin{aligned} \mathbf{c} \cdot \mathbf{b} &= (\mathbf{a} - \mathbf{r}) \cdot (\mathbf{a} + \mathbf{r}) \\ &= \mathbf{a}^2 + \mathbf{a} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{a} - \mathbf{r}^2 \\ &= \mathbf{a}^2 - \mathbf{r}^2 \\ &= 0\,, \end{aligned} \tag{12}$$

where the last step followed from (11). Thus we see a proof in Euclidean geometry that does not require coordinates, thus does not require special choices for coordinates to be chosen.

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