Projective Geometry 2: Dorwart's Introduction of Line Coordinates

P. Reany

May 10, 2022

Abstract

This paper examines Dorwart's introduction of Line Coordinates, and uses it to solve for the simultaneous tangent line to two conics.

1 Introduction

Here we examine the introduction of line coordinates by Harold L. Dorwart from his book [1], pp. 37–38. Athough Dorwart uses line coordinates in his proofs of some theorems in his book, I will not explicitly use them beyond this paper. But as the solved problems in this paper reveal, line coordinates do have a useful place in cartesian geometry.

The introduction of line coordinates is credited to J. Plücker.[1] Consider the equation for a line ℓ in the x, y-plane.

$$ux + vy + c = 0\,. \tag{1}$$

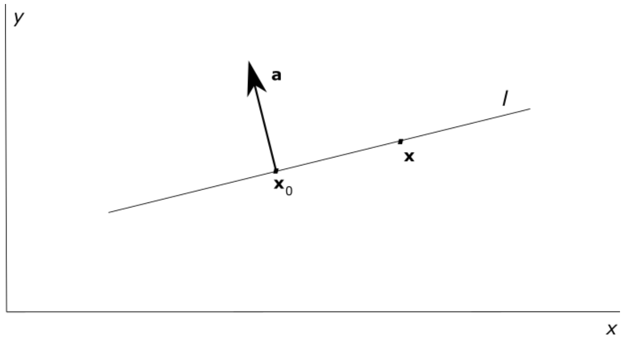

Figure 1. The geometry of line coordinates using Gibbs's dot product.

---

[1] J. Plücker (1801–68), is credited with others as introducing homogeneous coordinates, as well, in the early 1800's.

We usually think of the line equation (1) in coordinate geometry as having fixed 'coefficients' u and v. Following this constraint, we can think of a random point of the x, y-plane as being incident with ℓ if and only if its coordinates satisfy Eq. (1). Thus the two numbers u and v somehow represent line ℓ, though not uniquely, only up to parallelism.

In vector notation, $\mathbf{x}$ in the x, y-plane is incident with line ℓ if and only if, for some nonzero fixed vector $\mathbf{a}$ and some point $\mathbf{x}_0$ on ℓ,

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0\,. \tag{2}$$

(See Figure 1.) Setting $\mathbf{a} = (u, v)$, $\mathbf{x}_0 = (x_0, y_0)$, and expanding (2), we have

$$ux + vy - ux_0 - vy_0 = 0\,. \tag{3}$$

Now, setting $c = -ux - vy$, we get

$$ux_0 + vy_0 + c = 0\,. \tag{4}$$

We can think of this last equation as a prescription for all the lines through point (x_0, y_0) as we change the variables u and v; hence $[u, v]$ can be considered the 'line coordinates' of a line through a fixed point (x_0, y_0). Given $[u, v]$, one can calculate the slope of the line, so long as the line is not vertical. (By the way, the set of all lines through a point is called the *pencil* of lines through the point.)

However, there is more information in Eq. (1) than is required. Remember that line coordinates represent a direction in the x, y-plane, so all we need is their ratios to get that information. Thus, it won't hurt to divide (1) through by $c \neq 0$, to get

$$ux_0 + vy_0 + 1 = 0\,, \tag{5}$$

where we just replaced u/c by u and v/c by v.

Given Eq. (5), we can calculate the slope of the line. Let (x_1, y_1) and (x_2, y_2) be two distinct points on line (5). Then the slope of this line can be calculated as

$$\text{slope} = \frac{\Delta y}{\Delta x} = -\frac{u}{v}\,. \tag{6}$$

Problem 1

Dorwart gives us a challenge to find the equation of a tangent line to a parabola, using only lines coordinates and algebra, no calculus allowed. The equation for the chosen parabola is

$$2y^2 = x\,, \tag{7}$$

and its graph with arbitrary tangent line is given in Figure 2.

So how do we approach this problem? We need to find u and v. Our input information is x_0 and y_0 constrained by (7). So, we pick a y_0 and use (7) to calculate the x_0, and then combine all that with (5) to get a single equation in either x_0 and y_0. Replacing x_0 by $2y_0^2$ seems the easier way to go.

Thus, replacing x_0 by $2y_0^2$ in (5) yields a single equation in y_0, treating u and v as mere parameters:

$$2uy_0^2 + vy_0 + 1 = 0\,. \tag{8}$$

Employing the quadratic formula on this equation to solve for y_0, we have

$$y_0 = \frac{-v \pm \sqrt{v^2 - 8u}}{4u}\,. \tag{9}$$

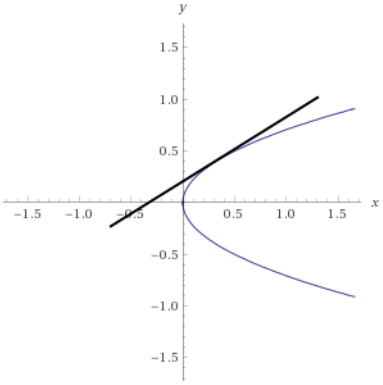

Figure 2. Parabola $2y^2 = x$ with tangent line, which intersect at point (x_0, y_0). (Graph fashioned with some help from WolframAlpha.com.)

Now, as every high-school algebra student knows, this equation admits three cases, depending on the value of the discriminant $v^2 - 8u$.

Case 1) If the discriminant is positive, then (9) admits two real values for y_0, implying that the line given by (8) intersects the parabola twice, and thus is not a tangent line.

Case 2) On the other hand, if the discriminant is negative, then (9) admits two complex values (more specifically, has none-zero imaginary part) for y_0, implying that the line given by (8) doesn't intersect the parabola at all.

Case 3) Finally, if the discriminant is zero,

$$v^2 - 8u = 0\,, \tag{10}$$

we have the Goldilocks condition that the line given by (8) intersects the parabola exactly once, and thus is a means for calculating the slope of the tangent line to the parabola. Eq. (10) is the *line equation* for the parabola.

Thus, to find the values of u and v in (5) to find the equation of the tangent, we substitute zero into the discriminant of (9) to get

$$v = -4y_0u\,, \tag{11}$$

and we're halfway there. Now, substituting this value for v into (10), we get

$$u = 2y_0^2\,, \tag{12}$$

Therefore the slope m of the tangent line

$$y = mx + b \tag{13}$$

is from (11)

$$m = -\frac{u}{v} = \frac{1}{4y_0}\,. \tag{14}$$

We can now solve for y-intercept b, to get

$$b = y_0 - mx_0 = y_0 - \frac{1}{4y_0}2y_0^2 = y_0/2\,. \tag{15}$$

And finally, the equation for the tangent line is

$$y = \frac{1}{4y_0}x + \frac{y_0}{2}\,. \tag{16}$$

Problem 2

Our last problem is to find the equations of the two lines of common tangency to the parabola $2y^2 = x$ (which we used in the last problem) and the ellipse (see Figure 3 for the graphic) given by the equation:

$$5x^2 + 20y^2 = 4\,. \tag{17}$$

We note now that the line equation we find for the ellipse must be the same as the line Equation (10) we already developed for the parabola because they are one and the same line of tangency.

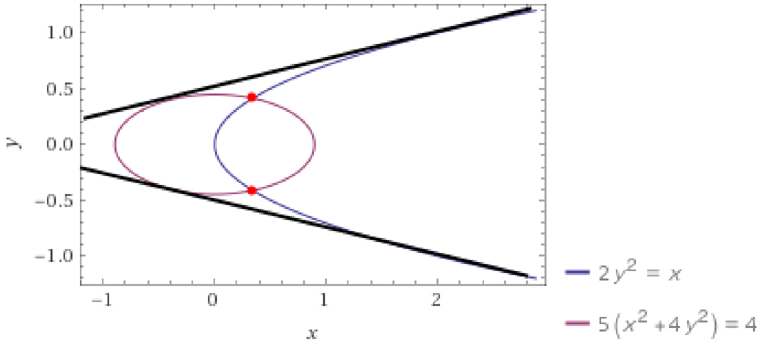

Figure 3. Problem: Find the equations of the common tangents of the parabola $2y^2 = x$ and the ellipse $5x^2 + 20y^2 = 4$. (Graph fashioned with some help from WolframAlpha.com.)

Our solution to this problem begins by performing steps on the ellipse that are similar to those we just performed on the parabola; namely, we will start with equation

$$ux_1 + vy_1 + 1 = 0\,, \tag{18}$$

and use the relations in (17) to eliminate x_1, say, in (18) and then have a quadratic in y_1, which we will use to extract the discriminant in variables u, v.

To get started, solve for ux_1 in (18) and then square both sides to get

$$u^2 x_1^2 = v^2 y_1^2 + 2vy_1 + 1\,. \tag{19}$$

Now use (17) to eliminate x_1 to arrive at this quadratic in y_1:

$$u^2(4 - 20y_1^2) = 5(v^2 y_1^2 + 2vy_1 + 1)\,, \tag{20}$$

or

$$(5v^2 + 20u^2)y_1^2 + 10vy_1 + (5 - 4u^2) = 0\,, \tag{21}$$

As we did in the case of the parabola, we set the discriminant of this quadratic in y_1 to zero, setting it equal to zero, to get the line equation

$$v^2 + 4u^2 = 5\,. \tag{22}$$

Now, to find the equations for the two tangent lines to these conics, we will need to solve (10) and (22) simultaneously, we get two roots for u, namely, $1/2$ and $-5/2$, of which the latter root is untenable. Therefore, we have that

$$u = 1/2 \quad \text{and} \quad v = \pm 2\,. \tag{23}$$

We can now write the equations of tangency for the two lines of fixed u, v and variable x, y as

$$\tfrac{1}{2}x \pm 2y + 1 = 0\,, \tag{24}$$

or, preferably as

$$x \pm 4y + 2 = 0\,. \tag{25}$$

References

[1] H. L. Dorwart, *The Geometry of Incidence*, Prentice-Hall (1966).