Projective Geometry 3: Dorwart's Proof of the Theorem of Pappus

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Abstract

This paper reviews a proof of Pappus's hexagonal theorem using homogeneous coordinates, as presented in the book *The Geometry of Incidence.*

1 Introduction

In his book *The Geometry of Incidence*, Harold L. Dorwart[1] presents his version of a homogeneous coordinate proof of Pappus's hexagon theorem. We re-produce that proof, adding a bit of detail that might make the proof easier to follow for the reader.

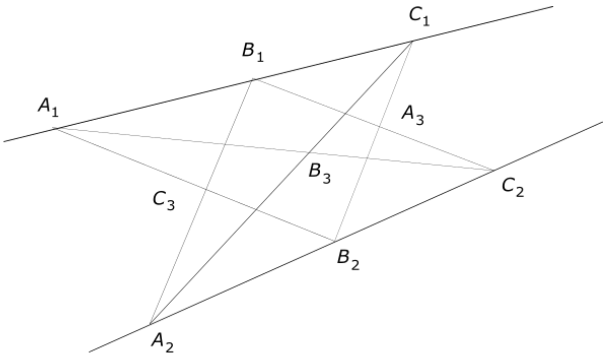

Figure 1. Pappus's hexagonal planar figure as presented by Dorwart, pp 82–84.

In the first paper of this series, we saw a proof of Pappus's hexagonal theorem using a coordinate-free vector approach. This paper will reveal a true coordinate-vector approach, using true homogeneous coordinates. We still have the same goal and the same constraints to employ against that goal, only this

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[1] *The Geometry of Incidence*, Prentice-Hall, 1966, pp 82-84.

time, all the shown points on the hexagon will be given coordinates, and calculations will be made using those coordinates.

Goal: Show that A_3, B_3, C_3 are collinear.

Constraints: A_1, B_1, C_1 are collinear, and A_2, B_2, C_2 are collinear.

The goal can be stated in vector form as, Show that

$$[A_3B_3C_3] = 0\,, \tag{1}$$

with constraints

$$[A_1B_1C_1] = 0\,, \tag{2a}$$
$$[A_2B_2C_2] = 0\,. \tag{2b}$$

2 Proof

Now, here's the advantage of using homogeneous coordinates: You get to choose the coordinates for a limited number of points (with convenient coordinates of 0's and/or 1's), in this case four points, no three of which are collinear. Dorwart chose to set

$$A_1 = (1,0,0)\,, \tag{3a}$$
$$C_1 = (1,1,1)\,, \tag{3b}$$
$$A_2 = (0,1,0)\,, \tag{3c}$$
$$A_3 = (0,0,1)\,. \tag{3d}$$

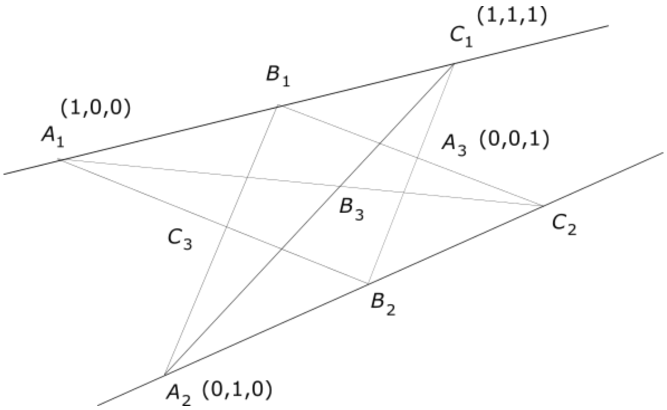

Figure 2. Pappus's hexagonal planar figure with specific assigned homogeneous coordinates, no three of which are collinear.

Obviously, assigning points with a lot of 1's and 0's will make computing triple scalar products, such as in (1), (2a), and (2b), a lot easier than using arbitrary values. So, now the updated figure becomes Figure 2 (above).

Terminology: I will refer to the points $A_1, B_1, C_1, A_2, B_2, C_2$ as the *basic six points of the figure*, or more simply, as the *basic six*.

Next, Dorwart assigns coordinates to three other points, consistent with the previous assignments he already made. (I call this procedure *propagating coordinates.*) On page 84, he tells us that his choices for points B_1 and B_2, respectively, are $(d, 1, 1)$ and $(1, 1, f)$, but he didn't tell us how he arrived at these coordinates. So let's do this now from basic principles.

We begin by assigning B_1 the coordinates (r, s, t) and use the linear constraint (via their triple scalar product) to place the necessary constraint on those coordinates to solve for them

$$[A_1 B_1 C_1] = 0\,, \tag{4a}$$

which expands to

$$\begin{vmatrix} 1 & 0 & 0 \\ r & s & t \\ 1 & 1 & 1 \end{vmatrix} = 0\,, \tag{4b}$$

yielding

$$1(s - t) = 0\,. \tag{5}$$

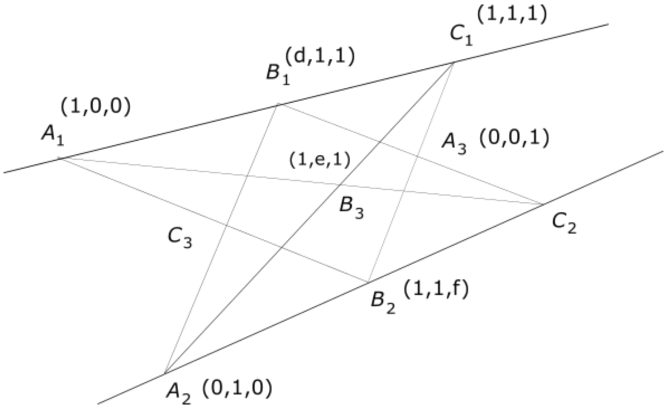

Figure 3. Pappus's hexagonal planar figure with specific additional homogeneous coordinates assigned, with only two points left to coordinatize.

In (5), we are free to choose s and t as any nonzero numbers, so long as their difference is zero. Dorwart choose $s = t = 1$. That leaves r. As Dorwart explained, it can be any real number except 1, since if it were 1, then B_1 would be the same as C_1, which is not allowed. Finally, to be consistent with Dorwart's notation, we set $r = d$. By similar reasoning and calculations, Dorwart chose $B_3 = (1, e, 1)$ with $e \neq 1$ and $B_2 = (1, 1, f)$ with $f \neq 1$. See Figure 3 on the previous page.

At this point in the development, I take a different approach than Dorwart's presentation. He introduced line coordinates to develop equations of lines in the plane in terms of variables x_1, x_2, x_3 (i.e., x, y, z). But I will show that this is unnecessary to attain the same goal.

Now we coordinatize points C_2 and C_3. We begin with C_2. Once again, we start with a general set of coordinates for this point: $C_2 = (r, s, t)$ and set $r, s \neq 0, 1$. Next we constrain these variables with the requirement that C_2 be collinear with A_2 and B_2. Therefore, we set

$$[A_2 B_2 C_2] = 0\,, \tag{6a}$$

which expands to

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & f \\ r & s & t \end{vmatrix} = 0\,, \tag{6b}$$

yielding

$$rf - t = 0\,. \tag{7}$$

From this last constraint we set $C_2 = (r, s, rf)$ and we'll use this value in our two remaining collinearizations to determine the needed constraints on values for d, e, f.

For starters, C_2 must also be collinear with A_1 and B_3. So,

$$[A_1 B_3 C_2] = 0\,, \tag{8a}$$

expands to

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & e & 1 \\ r & s & rf \end{vmatrix} = 0\,, \tag{8b}$$

yielding

$$erf - s = 0\,. \tag{9}$$

Lastly, C_2 must also be collinear with A_3 and B_1, So,

$$[A_3 B_1 C_2] = 0\,, \tag{10a}$$

expands to

$$\begin{vmatrix} 0 & 0 & 1 \\ d & 1 & 1 \\ r & s & rf \end{vmatrix} = 0\,, \tag{10b}$$

yielding

$$ds - r = 0\,. \tag{11}$$

So, using (9) and (11), we get the constraint for d, e, f:

$$def = 1\,. \tag{12}$$

And so now we move on to C_3.

Once we have coordinates for points C_3, we'll have enough information to evaluate Equation (1), yielding a constraint on the variables d, e, f. All we need now is to calculate the components of C_3 from its 'nearest neighbors' A_1, B_1, A_2, B_2 by use of C_3's *projective location*[2]:

$$C_3 = [B_1 A_2 A_1 B_2] \quad \text{(or in Gibbs's vectors)} \tag{13a}$$

$$C_3 = (B_1 \times A_2) \times (A_1 \times B_2)\,. \tag{13b}$$

Before we can calculate the triple scalar product of points A_3, B_3, C_3, we must first determine the homogeneous coordinates (i.e., 3-space coordinates) of point C_3. But this is easy to do. To use the notions of the two previous papers.

$$A_1 \times B_2 = (1,0,0) \times (1,1,f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & f \end{vmatrix} = (0, -f, 1)\,. \tag{14a}$$

Similarly,

$$B_1 \times A_2 = (d,1,1) \times (0,1,0) = (-1, 0, d)\,. \tag{14b}$$

So, substituting these last results into (13b), we have

$$C_3 = (df, 1, f)\,. \tag{15}$$

Calculating the scalar product of A_3, B_3, C_3, we get

$$[A_3 B_3 C_3] = \begin{vmatrix} 0 & 0 & 1 \\ 1 & e & 1 \\ df & 1 & f \end{vmatrix} = 1(1 - def) = 0\,, \tag{16}$$

where we used (12) So, we have proved that A_3 lies on the line containing B_3 and C_3.

3 Conclusion

Now that we've seen the solution to Pappus's hexagonal theorem in this paper and in Paper 1, we can see the huge difference in the approach to each proof. Sure, both methods of proof have embedded the projective plane into 3-space, while removing the origin from that plane. And each method uses Gibbs's vector algebra to formulate and solve constraints to reach the desired conclusion. But the first paper used truly coordinate-free vector methods, while this paper uses coordinate-laden vector methods. Furthermore, Dorwart employed equations of lines in the projective plane to help complete the proof, though my proof didn't require them. But, I still have a lot to learn about projective geometry, so I may be missing something important Dorwart knew that I don't.

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[2] The *projective location* of a point is given as the meet of two lines in the projective plane, which is represented by a triple cross product in the Gibbs's vector algebra.