Projective Geometry 7: Pascal's Theorem with Analytic Proof Adapted from Dorwart

P. Reany

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Abstract

This paper traces Dorwart's 'outline' of Pascal's Theorem using his analytic proof.

1 Introduction

We trace the 'outline' proof of Pascal's Theorem using a version of Harold L. Dorwart's proof from his book, *The Geometry of Incidence*, Prentice-Hall, 1966, pp. 107–111. I make some minor changes to the notation of Dorwart.

First, I'll quote Dorwart's statement of Pascal's Theorem:

> If the six (distinct) vertices of a hexagon lie on a non-degenerate conic, then the three points of intersection of pairs of opposite sides are collinear (on a line called the Pascal Line of the hexagon).

My conic of choice is an ellipse.

Up to this point in the book, we have only used straight lines and their points of intersections, but now we have conics, which are metrically defined. The circle is a set of all points equidistant from a center point. An ellipse is the set of all points, the sum of their combined distances from the foci are a constant. This a metrical nature of conics doesn't seem to bother Dorwart, and he doesn't address the issue at all.

What he does is to introduce a usable form of an ellipse equation on page 108:

$$x^2 + y^2 + z^2 + (r + 1/r)yz + (s + 1/s)xz + (t + 1/t)xy = 0\,. \tag{1}$$

Then he offers the six coordinates for points on the conics:

$$\begin{array}{lll} A = (0, r, -1) & C = (-1, 0, s) & E = (t, -1, 0) \\ B = (0, -1, r) & D = (s, 0, -1) & F = (-1, t, 0) \end{array}$$

and then he demonstrates that point A does actually satisfy Eq. (1).

Consider the figure below.

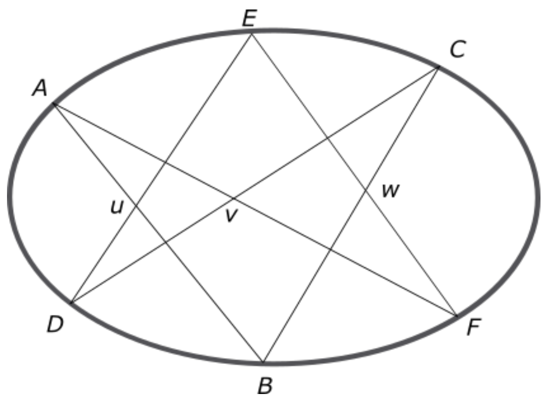

Figure 1. Pascal's Ellipse Configuration. This figure is an adaptation of the figure used in Dorwart's book on page 108. The theorem is that u, v, and w are collinear.

It's now a simple matter to calculate the projective locations of u, v, and w from these given points.

For example,

$$u = [EDAB]\,, \quad v = [CDAF]\,, \quad w = [CBEF]\,. \tag{2}$$

I'll show the steps only for point u, as the steps are similar for the other two points. Now,

$$u = [ED] \times [AB]\,, \tag{3}$$

where,

$$[ED] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & -1 & 0 \\ s & 0 & -1 \end{vmatrix} = \mathbf{i}(1) + \mathbf{j}(t) + \mathbf{k}(s) = (1, t, s)\,, \tag{4}$$

and

$$[AB] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & r & -1 \\ 0 & -1 & r \end{vmatrix} = \mathbf{i}(r^2 - 1) + \mathbf{j}(0) + \mathbf{k}(0) = (r^2 - 1, 0, 0)\,. \tag{5}$$

Therefore,

$$u = [EDAB] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & t & s \\ r^2 - 1 & 0 & 0 \end{vmatrix} = \mathbf{i}(0) + \mathbf{j}(s)(r^2-1) - \mathbf{k}(t)(r^2-1) = (r^2-1)(0, s, -t)\,. \tag{6}$$

Similarly,

$$v = [CDAF] = (s^2 r - r, 0, -s^2 t + t)\,, \tag{7}$$

$$w = [CBEF] = (t^2 - 1)(r, -s, 0)\,. \tag{8}$$

So

$$
\begin{aligned}
[uvw] &= (r^2-1)(t^2+1)\begin{vmatrix} 0 & s & -t \\ r & -s & 0 \\ s^2r-r & 0 & s^2t+t \end{vmatrix} \\
&= (r^2-1)(t^2+1)(0) \\
&= 0\,.
\end{aligned} \tag{9}
$$

Since the triple scalar product of the three points is zero, the points are collinear.