Projective Geometry 6: Pappus Two Concurrences and Grassmann-Plücker Relations

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Abstract

This paper proves the Pappus Two Concurrences lemma, and Grassmann-Plücker relations and related lemmas, in coordinate-free vector formalism.

1 Introduction

We continue the analytical methods developed in the last paper (i.e., Gibbs's vector algebra adapted to the projective plane) to prove that the extension of the Pappus hexagon, as shown in Figure 1, adding a point D of concurrence of the lines L_3, L_4, L_5 at point D, then line L_6 meets lines L_1 and L_2 at point a.

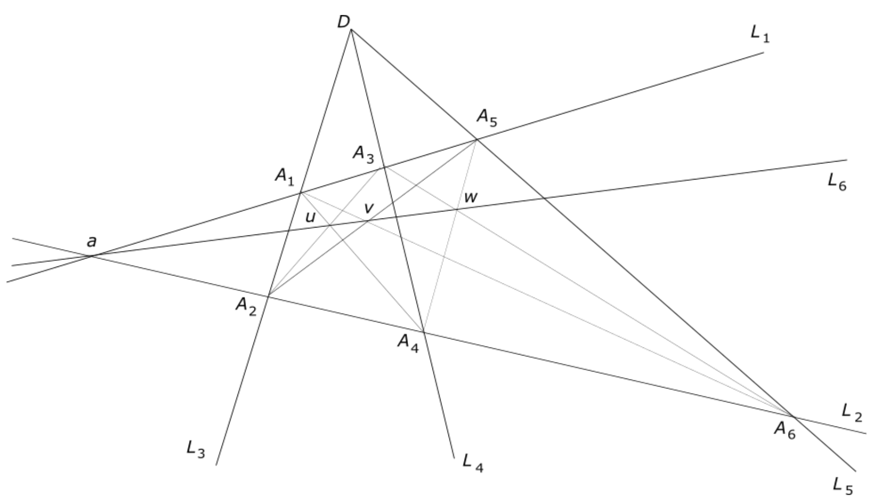

Figure 1. Pappus's hexagonal planar figure constructed so that lines L_3, L_4, and L_5 meet at point D.

For starters, points in this plane will be denoted algebraically either by a single letter or by the subscripts of letters bearing subscripts. For example, points a and D will be represented in their vector form as $[a]$ and $[D]$, and the points A_i by the vectors $[i]$. Also, the cross product of two vectors, such as A_1 and A_2, will be denoted by [12], and the 'triple' scalar product[1] of three vectors, such as A_1, A_3, and A_5, will be denoted (using the given order) by [135]. Written out explicitly

$$[135] = [1] \cdot [3] \times [5] = [1] \cdot ([3] \times [5]) = [A_1] \cdot ([A_3] \times [A_5])\,. \tag{1}$$

From the first paper we learned that three points are collinear if their triple scalar product is zero. For example, referring to the above figure, points A_1, A_3, and A_5 are collinear (according to the usual setup for Pappus's Hexagonal Theorem), and that means that

$$[135] = 0\,. \tag{2a}$$

Similarly, A_2, A_4, and A_6 are collinear, yielding

$$[246] = 0\,. \tag{2b}$$

It's the point of the Theorem of Pappus to show that as a result of the given construction that u, v, and w are collinear, giving us the additional constraint on the figure

$$[wuv] = 0\,. \tag{3}$$

Before going any further, I want to establish the meaning of point a. It is the meet of lines L_1 and L_2. Think of point a as being prior to drawing line L_6.

Now, it's obvious that with the additional construction of point D as the meet of lines L_3, L_4, and L_5, we've created a special version of Pappus's hexagon — so special, in fact, that L_6 is concurrent with L_1 and L_2 at point a. And this is what we need to prove.

I want to reiterate the relation of point a to line L_6: Yes, in the given figure, point a lies on L_6 because the graphics program I used forced it so; however, L_6 is defined, not by a, but by any two of the three points u, v, w, since they are collinear (this was the point of the original Pappus theorem we proved before in this series).

So, here's my plan on proving the current theorem: I want to show that, given (2a), (2b), and (3), and some constraints implied by the concurrence point D, that

$$[auv] = 0\,, \tag{4}$$

which, of course, means that point a is on line L_6. To keep from going in computational circles, we will express points a, u, v, w in terms of points A_1–A_6, which I'll refer to as the 'basic six'.

Now, I won't actually need the point D, explicitly, but as the point of concurrence of lines L_3, L_4, and L_5, it will provide additional constraints on the basic six that will be needed to complete the proof.

[1] My clumsy way of saying that I make a scalar out of three vectors.

2 Preliminary Stuff (A Review)

Let a, b, c, d be distinct point in a projective plane. Let a and b be points on one line, and c and d be points on another line, then their meet point p, say, is given algebraically by

$$p = [abcd] \equiv [ab] \times [cd] = (a \times b) \times (c \times d)\,, \tag{5}$$

where, for calculational purposes, a, b, c, d are the tips of vectors in three-space, whose bases are at the origin, which is not on the plane of the figure. (This is all explained in full in the first paper.) Also, I think of $[ab]$, for example, as a free vector in the plane,[2] going from base point b to end point a. This way, I can define meet points from point pairs in a consistent way, or else I can get the sign between terms to get off. So, by the foregoing procedure, the points a, u, v, w will be expressed in terms of the basic six in a sign-consistent manner.

Definition: I refer to the location of a point p in the projective plane, when it is expressed in the form $p = [abcd]$, as the *projective location* of the point.

Case in point: Look again at (3) and (4). There is a 'simple' equation that relates $[a]$ and $[w]$ (which will be derived soon), namely,

$$[a] = [4][653] + [3][654] - [w]\,. \tag{6}$$

Now, 'hitting both sides' (dotting through on both sides) by the vector $[uv]$ on the right (though, on the left works, too, because the the dot product is commutative), we get

$$[auv] = [4uv][653] + [3uv][654]\,, \tag{7}$$

where we have used the constraint $[wuv] = 0$.[3] Clearly, $[auv]$ will vanish if the two terms on the RHS of this last equation cancel each other out, and this requires that we assign the signs of points derived from (5) consistently.

We are about to begin a series of easy but protracted calculations to bring the given constraints into usable form, and to do this, the reader needs to remember that the triple scalar product is a determinant, or at least that it acts like one. For example, the triple scalar product of vectors a, b, and c (in that order) is

$$[abc] = [cab] = [bca]\,, \tag{8}$$

just like cyclically permuting the rows of a determinant. Similarly,

$$[abc] = -[bac]\,, \tag{9}$$

and similarly for any odd number of transpositions of letters a, b, and c.

[2]This makes sense as $[ab]$ is, in the projective sense, the join of points a and b (that is, the line connecting those two points) in the projective plane.

[3]Obviously we must use the constraints to solve the problem, and it was to use this particular constraint that I chose to construct Equation (6) in the first place.

This brings us to another convention that will also help to get the signs of terms correct. In any 'final' result containing triple scalar products of the basic six points, we will display the products in order of ascending magnitude (i.e., numbers before letters and letters in alphabetical order). To demonstrate, we rewrite (6) as

$$-[a] = [4][356] + [3][456] + [w]\,, \tag{10}$$

and (7) as

$$-[auv] = [4uv][356] + [3uv][456]\,. \tag{11}$$

Once again, we are finished if we can show that the two terms on the RHS of (11) cancel each other out.

We now begin a series of useful lemmas. The ones without proof were proved in the first paper.

Lemma 1 (on the meet of distinct lines)
Let A, B, C, D be distinct points in the projective plane. Then

$$[ABCD] = B[ACD] - A[BCD] \tag{12a}$$
$$= C[ABD] - D[ABC]\,. \tag{12b}$$

Proof: We begin by converting the bracket expressions into standard Gibbs's vector expressions and proceed from there:

$$[ABCD] = (A \times B) \times (C \times D)\,. \tag{13}$$

Using standard Gibbs's vector algebra,[4] we can prove (treating $A\times B$ as a single vector) results in the general formula

$$\begin{aligned}[ABCD] &= [C](A \times B) \cdot D - [D](A \times B) \cdot C \\ &= C[ABD] - D[ABC]\,.\end{aligned}$$

Now, to prove (12a), we have that

$$\begin{aligned}[ABCD] &= [(AB)(CD)] = -[(CD)(AB)] = -[CDAB] \\ &= -\big(A[CDB] - B[CDA]\big) \\ &= B[CDA] - A[CDB] \\ &= B[ACD] - A[BCD]\,. \end{aligned} \tag{14}$$

And similarly prove (12b) as

$$\begin{aligned}[ABCD] &= [BADC] = [D](B \times A) \cdot C - [C](B \times A) \cdot D \\ &= D[BAC] - C[BAD] \\ &= C[ABD] - D[ABC]\,,\end{aligned}$$

[4] We expand by using $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\mathbf{a}\cdot\mathbf{c} - \mathbf{c}\mathbf{a}\cdot\mathbf{b}$ of a double cross product, where $\mathbf{a} = A\times B$, $\mathbf{b} = C$, and $\mathbf{c}$ =D.

where the change in signs from the second to final step occurred because we changed $[BAC] \to [ABC]$ and $[BAD] \to [ABD]$.

Lemma 2 (on transposing the center points of $[ABCD]$)
As is often the case in mathematics, one is interested in the effect of transposing adjacent noncommuting factors in an expression. Let's see what the cost is when we transpose elements B and C in $[ABCD]$:

$$[ABCD] = B[ACD] + C[ABD] - [ACBD]\,. \tag{15}$$

The proof of this result is not difficult and is left to the interested reader.

3 The Main Result

So, what are the projective locations of points a, u, v, and w? We consult the figure on page 1 and conclude the following locations:

$$[a] = [6453]\,, \tag{16a}$$
$$[w] = [6354]\,, \tag{16b}$$
$$[u] = [3214]\,, \tag{16c}$$
$$[v] = [5216]\,. \tag{16d}$$

Let's now employ Equation (15) to establish (6). The most direct way to do this (noting that $[6453] = -[6435]$) is to recast (16a) as (the proof following)

$$\begin{aligned} -[a] &= [6435] \\ &= [4][635] + [3][645] - [6345] \\ &= [4][356] + [3][456] + [6354] \\ &= [4][356] + [3][456] + [w]\,, \end{aligned} \tag{17}$$

which establishes (10).

Now, back to Equation (11):

$$-[auv] = [4uv][356]{+}[3uv][456]\,. \tag{11}$$

I'm going to regard $[4uv]$ and $[3uv]$ as, respectively, $[4] \cdot [uv]$ and $[3] \cdot [uv]$, so I need an appropriate expression for $[uv]$. Using (16c) and (16d), we get

$$[uv] = [3214] \times [5216]\,. \tag{18}$$

Using Equations (12a) or (12b), each factor on the RHS of this last equation can be expanded as differences of vectors in many ways. I'll tailor this cross product appropriate for its dot product with the specified vector, either [3] or [4].

To dot $[uv]$ by [4], I'll expand $[uv]$, using (12b), as

$$\begin{aligned}[uv] &= [3214] \times [5216] \\ &= ([1][324] - [4][321]) \times ([1][526] - [6][521]) \\ &= -[16][324[]521] - [41][321][526] + [46][321][521]\,, \end{aligned} \tag{19}$$

where we have used that for arbitrary vectors a and b, $[aab] = [aba] = 0$. In which case, we have

$$[4uv] = -[416][324][521] = [146][234][125]\,, \tag{20}$$

And, to dot $[uv]$ by [3], I'll expand $[uv]$, using (12a), as

$$\begin{aligned}[uv] &= [3214] \times [5216] \\ &= ([2][314] - [3][214]) \times ([2][516] - [5][216]) \\ &= -[25][314][216] - [32][214][516] + [35][214][216])\,, \end{aligned} \tag{21}$$

in which case, we have

$$[3uv] = -[325][314][216] = [235][134][126]\,. \tag{22}$$

So, on substituting (20) and (22) into (11), we get

$$-[auv] = [146][234][125][356] + [235][134][126][456]\,. \tag{23}$$

Yes, the RHS of (23) looks a mess, but we're really not that far away from the solution we seek. We have yet to employ the constraints on the basic six imposed by the concurrence of lines L_3, L_4, and L_5 at point D. Based on those constraints, it will shortly be shown that

$$[146][234] = [456][126]\,, \tag{24a}$$
$$[124][234] = -[245][126]\,, \tag{24b}$$
$$[126][234] = [126][256]\,, \tag{24c}$$
$$[123][134] = [136][125]\,, \tag{24d}$$
$$[235][134] = -[356][125]\,, \tag{24e}$$

and there are many more that could be listed.

Now, using (24a) in (23), we have that

$$\begin{aligned}-[auv] &= [456][126][125][356] + [235][134][126][456] \\ &= [456][126]\big([125][356] + [235][134]\big)\,. \end{aligned} \tag{25}$$

And the expression inside the parentheses of this last equation is shown to be zero because of (24e). Therefore, we have indeed proven that $[auv] = 0$, proving that point a is on line L_6. I guess my graphics program got it right, after all.

4 Lemmas Based on Concurrence at D

The projective location of D can be given in three different, but related, forms:

$$D = [1234]\,, \tag{26a}$$
$$D = [1256]\,, \tag{26b}$$
$$D = [3456]\,, \tag{26c}$$

from which we derive the three related equations:

$$[1234] = [1256]\,, \tag{27a}$$
$$[1234] = [3456]\,, \tag{27b}$$
$$[1256] = [3456]\,. \tag{27c}$$

The first kind of relations we can extract from these is of the form $[xyz] = [rst]$. We won't need them in this paper, but I'll show how to extract one of them to serve of an example of how to do it.

Start with (27a), say, and expand both sides by the left ends of the brackets, like such

$$[2][134] - [1][234] = [2][156] - [1][256]\,. \tag{28a}$$

First, multiply through by [46] to get

$$[146][234] = [146][256]\,, \tag{28b}$$

where, once again, we used that $[246] = 0$. Now we divide through by [146] to get[5]

$$[234] = [256]\,. \tag{28c}$$

Many more such simple relations can be derived this way. But for our current needs, we will from now on expand the equation to have four different vectors showing in each equation.

Now, on to the proofs of the list of identities we will be using (though there is no need to prove them all). For our first example, expand (**??**), as follows

$$[2][134] - [1][234] = [5][126] - [6][125]\,. \tag{29a}$$

Again, multiply through by [46] to get (again, using that $[246] = 0$)

$$-[146][234] = [546][126]\,, \tag{29b}$$

which can be writen in canonical form as

$$[146][234] = [456][126]\,. \tag{29c}$$

And this last equation is the same as (24a).

[5]Of course [146] is not equal to zero; only the scalar products of points all on either L_1 or L_2 is zero.

For the last example in this section, I'll establish (24e). To that end, hit (29a) by [35] (i.e., use a dot product), to get

$$[235][134] = -[635][125]\,, \tag{30a}$$

where we used that $[135] = 0$. And writing this in canonical form, we get

$$[235][134] = -[356][125]\,. \tag{30b}$$

5 The Grassmann-Plücker Relations

I throw this into the mix because it's a trivial exercise for the reader to follow and it appears in some treatments of projective geometry.

Lemma: For any five points in the projective plane, a, b, c, d, e, the following relationship holds:

$$[abc][ade] - [abd][ace] + [abe][acd] = 0\,. \tag{31}$$

We've seen equations like this before. Each term is the product of two scalars (triple-vector scalar product).[6] The obvious place to start trying is with the following form

$$[wu] \cdot [txyz]_1 = [wu] \cdot [txyz]_2\,, \tag{32}$$

where the subscripts indicate that we intend to expand the point location $[txyz]$ in two different ways, as we have done before. So, let's try it with

$$[ae] \cdot [cbad]_1 = [ae] \cdot [cbad]_2\,, \tag{33}$$

to get

$$[ae] \cdot ([b][cad] - [c][bad]) = [ae] \cdot ([a][cbd] - [d][cba])\,, \tag{34}$$

which expands to

$$[aeb][cad] - [aec][bad] = -[aed][cba]\,. \tag{35}$$

From here we just collect all terms and follow our rule of canonical ordering of the letters in the brackets.

$$[abe][acd] - [ace][abd] + [ade][abc] = 0\,. \tag{36}$$

We can merely reorder the terms to get (31).

In his online PDF paper on the theorem of Pappus, Richter-Gebert Jürgen (Technische Universität München) uses the Grassmann-Plücker Relations as one of his nine proofs:

[6] Clearly, we should think of this relation as some sort of identity, since it must hold for arbitrary choice of the five points.

https://www-m10.ma.tum.de/foswiki/pub/Lehre/
WS0809/GeometrieKalkueleWS0809/ch1.pdf
section 1.3

or

https://www.researchgate.net/publication/226712690_Pappos
%27s_Theorem_Nine_Proofs_and_Three_Variations

from his book *Perspectives on projective geometry. A guided tour through real and complex geometry*.

More on this is found in Chapter 6:

https://www-m10.ma.tum.de/foswiki/pub/Lehre/
WS0809/GeometrieKalkueleWS0809/ch6.pdf
section 6.5