Projective Geometry 1: Theorem of Pappus and Introduction to the Series

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Abstract

This paper quickly introduces the barest rudiments of projective geometry and then homogenous coordinates in order to present a proof to the Theorem of Pappus. **Prerequisites:** a bit of linear algebra (how to evaluate determinants of 2×2 and 3×3 matrices) and Gibbs's vector algebra, using the dot and cross products.

1 Introduction to the Series

This series of short papers seeks to present the solutions to significant early problems or theorems in projective geometry (involving only the real projective plane), one in each paper.[1] This paper, being of an introductory nature, must introduce concepts and notations common to all the papers, and thus is the longest of them all — at least as of this writing.

Over the last few decades, I have tried on numerous occasions to learn projective geometry, only to give up each time, convinced that my brain could not absorb such strange concepts or that the presentation of the material is just too garbled in translation.

I offer a quotation from an article by Evelyn Lamb, who referenced mathematician Emille Davie Lawrence [2]

> The projective plane is one of the hardest "basic" mathematical objects for me to imagine I'm calling it basic, even though it's not one of the first mathematical objects you meet, because it's one of the building blocks for other surfaces, as Emille Davie Lawrence mentioned when we talked to her for the My Favorite Theorem podcast.

But, when at last, I finally made some headway into the subject, I discovered that the subject is simpler than I had thought and that it could be presented

[1] Perhaps I should refer to this subject as 'naive projective geometry', because I am informed by topologists that the real projective plane cannot be embedded into $\mathbb{R}^3$, which I intend to do.

[2] https://blogs.scientificamerican.com/roots-of-unity/a-few-of-my-favorite-spaces-the-real-projective-plane.

easier than I had found it. This series is my attempt at doing so. I call the overall series of papers: *Projective Geometry for the Impatient,* by which I mean that there is a lot of projective geometry I will skip over to get right into proving the important early theorems, like Pappus's Hexagon theorem, Desargue's theorem, Pascal's Theorem, the invariance of the cross ratio, and, lastly, the finite projective geometry of order n with Fano's figure and its automorphisms. This series will make the subject even simpler than the full subject by only considering the projective plane. Perhaps someday I will venture into higher-dimensional projective geometry. By the way, projective geometry can be formed over any field, but for this series, it will be over the real field only.

We all have some intuition about what constitutes the subject matter of geometry: At a minimum, it contains relationships among points, lines, planes, conics, etc. In the ancient Greek days of the subject, Euclidean geometry was the main emphasis. It allowed for constructions with ruler[3] (straight-edge with rulings on it) and compass. It was a manifestly metrical study of geometry, whose interests are distances, angles, betweenness, and perpendicularity, all of which are lost in projective geometry, which cares more about incidence relations and invariant properties of figures under collineations, that is, bijections between projective spaces that preserve incidence relations.

So, how do we go from Euclidean geometry to projective geometry? Well, first we have to pass through the affine geometry, which starts at Euclidean geometry and drops the use of compass and ruled straight-edge, and allows drawing with only an unruled straight-edge. One can be forgiven for wondering if this stripped-down version of geometry has anything of importance to state or prove, but it does.

Anyway, even *this* is not quite projective geometry. In projective geometry, we intentionally build into this new geometry a duality, uncommon in geometries, in which 'point' and 'line' can be simultaneously swapped in any given axiom or true theorem and the result is another true theorem. Perhaps this is facilitated by the fact that these terms are left undefined in the first place, so no one knows what they 'really' mean, anyway, and the best one can do is to invent a geometric model that fits the axioms of the geometry at hand.

Beginning with the axiom that *For any two points in the plane, there exists a unique line joining them*, we must then be able to claim that *For any two lines in the plane, there exists a unique point incident with both of them.* Now, this is a strange notion in both Euclidean and affine geometries, when it comes to parallel lines in the plane. Surely, they never meet, so they will not have any points in common. But in order to enforce duality, projective geometry takes that 'one step beyond' and claims that all parallel lines meet at infinity! (Well, just how would you prove *that* wrong?!) So, the family of all parallel lines in a plane in a given 'direction' meet at a unique *ideal* point at infinity.[4] And there

[3]Purists might claim that Euclidean geometry does not allow rulers, per se, but only unmarked straight-edges to draw with, but since by use of a compass, marks of equidistance can be made on paper, I don't see much difference.

[4]This notion of 'direction' seems to be extrinsic to projective geometry, but I don't see how to make real progress with an algebraic form of it without it.

is a unique point at infinity for each direction in the plane, for a total of one ideal line at infinity containing all the ideal points at infinity.[5]

We may as well now attempt to deal with the question of the set theoretic notion of line and point: Are lines in projective geometry the union of points? In other words, are lines made out of points? Well, the ideal line seems to be made out of ideal points, but is a similar notion true for the affine lines in projective geometry? Perhaps not. At least I see no reason to believe so. However, when we get to the homogeneous coordinate model for the projective plane, it might seem so.[6] Nevertheless, for sure, when we get to finite geometry (at the end of this series), we'll see that a line can be incident with only a finite number of points, it then makes no sense to think of a line as the continuum of points we used to draw it.

Before the invention of the homogeneous coordinate model of projective geometry, proofs in the subject were basically dependent on the methods of proof in Euclidean geometry, which, based on the analysis just presented on the non-metrical nature of the subject, doesn't seem appropriate. For example, in the various Euclidean proofs of Pappus's hexagonal theorem, one is likely to see the use of such standard synthetic theorems as that of Ceva and Menelaus.

Okay, just what do I mean by 'synthetic'? I'll let Wikipedia answer this question:

> Synthetic geometry (sometimes referred to as axiomatic or even pure geometry) is the study of geometry without the use of coordinates or formulas. It relies on the axiomatic method and the tools directly related to them, that is, compass and straightedge, to draw conclusions and solve problems.

Thus, the so-called 'axiomatic' approach to geometry is that of classical Euclidean geometry, and it exists in contrast to the 'newer' form of geometry based on coordinates, which was the gift to us from René Descartes (1596–1650), and it is also known as 'analytic geometry'. Perhaps these contrasting terms arose in the description of geometry to emphasize the apparent synthetic/analytic dichotomy of geometry in the nineteenth century. In any case, calling Euclid's geometry 'axiomatic' as though coordinate geometry is not, is certainly confusing. Surely cartesian geometry has rules (axioms) by which it operates.

However, today, being a century and a half away from nineteenth century geometry, the state of the art of geometry has progressed remarkably. First,

[5] Projective geometry is to me one of the best reasons to believe the rule: You don't just invent the math that's true; you also invent that math you need to solve problems. I know of an outspoken mathematician who decries that lack of proper foundation to modern mathematics because, as he sees it, it allows for actual, rather than merely potential, infinity. Yet, he warmly praises projective geometry with its ideal points at actual infinity. For, one cannot imagine the intersection of two lines to meet at a point at 'potential infinity' and give it any logical meaning other than at a 'finite distance', in which case their intersection is at a 'real' point. My point being that an overly conservative fear of the invention of mathematics that goes 'one step beyond' the demonstrably true is bound to stymy the growth of mathematics in the name of certainty.

[6] We will see in the next paper in this series that the homogeneous model admits the parameterization of lines by continuous variables.

to Gibbs's vector algebra and second the extension theory of Grassmann, and third to the Geometric Algebra of D. Hestenes ([3], [4], [5]) and others. Viewed from the modern perspective, the attempt to think of geometry as a mere dichotomy between figures embedded in coordinate-free Euclidean space or figures embedded in cartesian space as a bit naive.

Anyway, it was just a matter of time, I suppose, until someone applied the methods of analytic (i.e., coordinate) geometry to the projective plane, and, voila, homogeneous coordinate model of projective geometry was the result. I give a more detailed description of this model later on in this paper, including how the 'homogeneous' part of the title comes in. Suffice to say for the time being that the projective plane arises by embedding the affine plane in $\mathbb{R}^3$, identifying it as the plane $z = 1$. Then, the ideal points become the points in the x, y-plane. In other words, the former points at infinity become points 'close by', but only in a different (i.e., parallel) dimension than the affine plane itself.

So what mathematician/s should we credit for the invention of the homogeneous coordinate model of projective geometry? According to the renowned geometer H. S. M. Coxeter [1] (p. 4):

> The last vestiges of dependence on ordinary geometry were removed in 1871, when Felix Klein provided an algebraic foundation for projective geometry in terms of "homogeneous coordinates," which had been discovered independently by K. W. Feuerbach and A. F. Möbius in 1827.

Once again we run into a confusing statement. It's not as though one cannot do algebra in synthetic geometry. For example, one could be given three of four measures of the sides of two similar figures and asked (by proportionality) to set up an algebraic equation to solve for the unknown length. What 'algebraic foundation' means is that the merely named points (usually be assigning them letters) of synthetic geometry are replaced by coordinatized points in the analytic geometric description of geometry: In other words, They're given you a number, and takin' way your name, so to speak. Nevertheless, we don't (always) need coordinates to apply algebra to geometry

The beauty of the homogeneous coordinate method is that it allows us to choose four canonical points to assign to the vertices of figures in the projective plane: $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$,[7] and then use these to assign coordinates to all other relevant points in the figure. Then, using what appears to be simple, but not so motivated methods of linear algebra[8] (such as the product of a row matrix and a column matrix and the evaluation of 2×2 and 3×3 matrices), one has a ready-made method to determine if three points are collinear or not.

[7]Note that the first three of these points are ideal (i.e., in the x, y-plane). Their benefit is that we will be doing dot and cross products with these vectors that are conveniently made up only 0's and 1's.

[8]It seems to me that just claiming that the homogeneous coordinate model is really Gibbs's vector algebra clarifies the geometric interpretation of this system of operations.

But when I first saw how this homogeneous coordinate model was being setup to solve Pappus's hexagonal theorem, even before I went through the proof in H. L. Dorwart's book *The Geometry of Incidence* [2] using that system, I said to myself something like this: "Hey, this is just Gibbs's vector algebra! I can solve this theorem myself using coordinate-free vector methods!" And so I did, and the central result of this paper is the presentation of that proof.

In my proof, I arbitrarily set the projective plane in $\mathbb{R}^3$, except that it does *not* contain the origin. However, I merely labeled the important vertices but did *not* assign coordinates to them. Then I applied the dot and cross products available in Gibbs's vector algebra and eventually found a proof. Only after that did I learn Dorwart's proof, which uses coordinates.

Having done the proof in both systems, I find that each system has advantages and disadvantages over the other. I go into these differences later on in the series. In summary, both methods are just the Gibbs's vector algebra. The homogeneous coordinate version assigns coordinates to points of the figures; the other version does not. The question to answer now is, What name shall we employ to label this 'other' method? How about the "coordinate-free Gibbs's model"?

2 Introduction: Relating points in the projective plane to vectors in $\mathbb{R}^3$

The subject of projective geometry is vast. The purpose of this paper is to introduce only enough rudiments of the subject so as to get a feel for the subject and then to use this information to solve a real problem, namely, the Theorem of Pappus, also known as Pappus's hexagon theorem, and similar problems in later papers. More foundational proofs will be given in later papers when the topic of finite geometries comes up.

Pappus of Alexandria is described as one of the last great mathematicians of antiquity. Wikipedia claims that we can estimate the general time of his writing to be about 320 AD, inferred from a notation Pappus made about a solar eclipse reckoned to be on 18 October 320.[9]

Now, in the abstract I promised to introduce homogenous coordinates, and I will, but only to connect their properties to Gibbs's vector algebra. What makes projective geometry work is a one-to-one correspondence in 3-space between planes and lines on the one hand and lines and points, respectively, on the other.

Here's the basic setup:

1) Take the planar figure you want to analyze (that naturally sits in $\mathbb{R}^2$) and embed it into $\mathbb{R}^3$ such that the origin $(0, 0, 0)$ is not in the plane. This structure is referred to as $\mathbb{RP}^2$. Let's choose a particular projective plane and call it Σ.

[9] The sun, moon, and stars were given for reckoning the times and seasons, and apparently the eclipses for estimating the year of events in antiquity before accurate calendars and record keeping were available, if that leaves us any the wiser in this case.

2) Note that there is a 1-1 correspondence between lines through the origin and points on Σ (i.e., where they intersect).

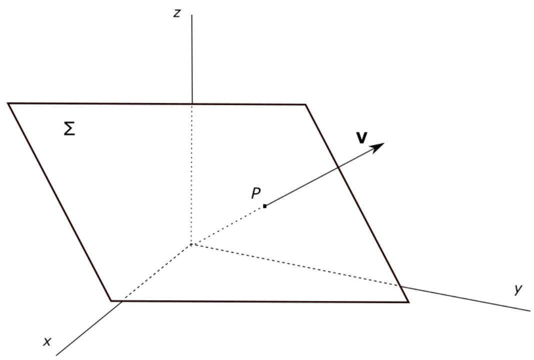

Figure 1. The 1-1 correspondence between lines through the origin, in this case represented by the vector $\mathbf{V}$, and points on the plane Σ. The intersection point of the line represented by the vector $\mathbf{V}$ and the plane Σ is the point P. What's important about the vector is not its length, but its direction in 3-space.

Now, it turns out that for projective geometry we don't need to describe the points in Σ as unique vectors from the origin to Σ. All that is needed is to find any vector along the line connecting the origin and the point in Σ. This will be proved later on.

Also note that there is a 1-1 correspondence between planes that contain the origin (which I refer to as *oriplanes*[10]) and lines in Σ. Thus, to test if three points in Σ are collinear, all we have to do is prove that the three points each correspond to three lines in some oriplane, which is easy to do with Gibbs's vector algebra. Why is this? Because, if the three points lie simultaneously on two distinct planes, they must lie on the intersection of the two planes, which is a line.

[10]Of course, our oriplanes have no resemblance to the Japanese origami planes.

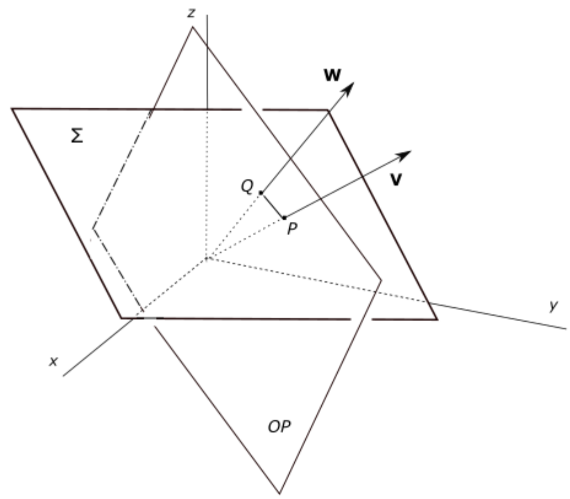

Figure 2. The 1-1 correspondence between lines through the origin, in this case represented by vectors $\mathbf{W}$, $\mathbf{V}$, and points Q, P, respectively, on the (projective) plane Σ. The oriplane OP is the unique plane through the origin containing the 'lines' $\mathbf{W}$ and $\mathbf{V}$. The orientation in 3-space of the plane OP can be represented by the normal vector $\mathbf{N}$ to the plane OP, given by $\mathbf{N} = \mathbf{V} \times \mathbf{W}$. What's important about this vector is not its length, but its direction in 3-space.

So, what is the secret to this mysterious replacement of lines in the plane $\mathbb{R}^2$ with homogeneous coordinates in $\mathbb{R}^3$? Simply this: The generic line in $\mathbb{R}^2$ with standard (non-homogenous) form

$$Ax + By + C = 0\,, \tag{1}$$

has been replaced by the line in $\mathbb{RP}^2$ Σ, with generic homogeneous form

$$Ax + By + Cz = 0\,. \tag{2}$$

So, how is this done? Imagine we have an oriplane OP with its normal vector $\mathbf{N}$ having its base point at the origin. Imagine also that we have any other nonzero vector, say $\mathbf{X}$, in the same plane with its base point at the origin. Then the dot product of these two vectors is zero, for they are orthogonal to each other, i.e.,

$$\mathbf{N} \cdot \mathbf{X} = 0\,. \tag{3}$$

Now, imagine parallel transport of $\mathbf{X}$ to any other place in OP, and call it $\mathbf{X}'$. The wonderful thing is that by parallel transporting $\mathbf{X}$ from the origin, we have to move its base point and its end point by the same amount, call it $\mathbf{D}$. Therefore, with $\mathbf{0} = (0, 0, 0)$ and $\mathbf{X} = \mathbf{X} - \mathbf{0}$,

$$\mathbf{X}' = (\mathbf{X} + \mathbf{D}) - (\mathbf{0} + \mathbf{D})\,, \tag{4}$$

where, because of the translation away from the base point at the origin, $\mathbf{X}'$ is no longer a 'proper' vector with coordinates, but is now an affine vector with components. By a 'proper' vector I mean a vector whose base point is always at the origin. An affine vector can have its base point anywhere. So, if we dot this affine vector by $\mathbf{N}$, we get

$$\begin{aligned}\mathbf{N}\cdot\mathbf{X}' &= \mathbf{N}\cdot(\mathbf{X}+\mathbf{D}) - \mathbf{N}\cdot(\mathbf{0}+\mathbf{D}) \\ &= \mathbf{N}\cdot\mathbf{X}+\mathbf{N}\cdot\mathbf{D}-\mathbf{N}\cdot\mathbf{0}-\mathbf{N}\cdot\mathbf{D} \\ &= \mathbf{N}\cdot\mathbf{X} = 0\,. \end{aligned} \tag{5}$$

So, what we've shown is that the normal to a plane is orthogonal to every vector in the plane, and to every line in the plane, and to every line segment in the plane.

Going back to Figure 2, let us regard the vector $\mathbf{X}$ as the vector going from point P to point Q, and let it have components

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\,. \tag{6}$$

Now let $\mathbf{N}$ be the normal to oriplane OP,[11] having coordinates (or components, since it doesn't matter in this case)

$$\mathbf{N} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}\,. \tag{7}$$

The components of vector $\mathbf{X}$ have, so far, only one constraint on them, given by Eq. (3), therefore,

$$\mathbf{N}\cdot\mathbf{x} = [A, B, C]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\,, \tag{8}$$

which gives us Eq. (2). Thus, Eq. (2) represents a line in the oriplane OP that goes through the origin and is parallel to line segment $\overline{QP}$.

We're now at the point where we can set down the two building blocks of incidence geometry from the projective geometry viewpoint:

1) Given three points in the projective plane, do we have a simple test to determine if they are incident with one line?

2) Given that all distinct pairs of lines in the projective plane intersect in a point, do we have a means of representing that intersection point in terms of the points on the two lines?

[11] Actually, we can rescale this normal by any nonzero factor and, for the purposes of projective geometry, it's still 'the' normal.

We'll deal with the first question now.

Figure 3 is an extension of Figure 2 by adding the named point R to Σ. It looks like it lies on the same line that contains points P and Q, but we need a test for this. These three points are collinear if and only if they lie on the line of intersection of planes OP and Σ.

Now we make a critical simplification. We're going to treat the points P, Q, and R as the tips of vectors in $\mathbb{R}^3$ with base points at the origin $\mathbf{0}$.

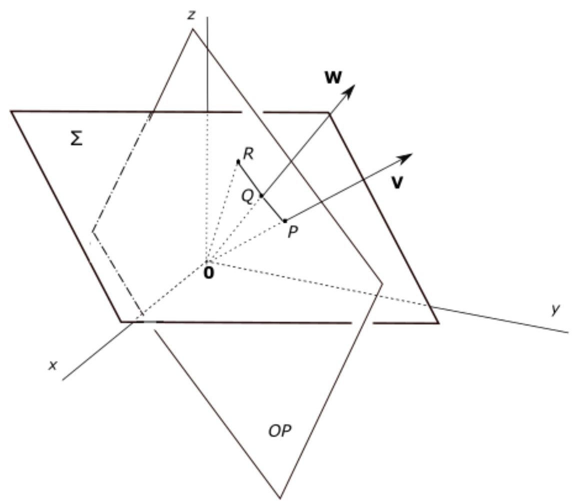

Figure 3. Compared to Figure 2, we've added the point R in the projective plane Σ. We look for a means to determine algebraically if the three points P, Q, and R are collinear.

If points P, Q, and R are collinear, then vectors P, Q, and R lie in the same oriplane, namely OP, in which case, everyone of those vectors is orthogonal to any normal to the plane. We can find a normal vector to OP by taking the cross product of any two of those three vectors, such as $R \times Q$ or $R \times P$ or $P \times Q$, etc.[12] Then we have that

$$P \cdot (R \times Q) = Q \cdot (R \times P) = R \cdot (P \times Q) = 0\,. \tag{9}$$

It may seem at this point of the discussion a bit odd not to use boldface for vectors, such as

$$\mathbf{P} \cdot (\mathbf{R} \times \mathbf{Q}) = \mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = \mathbf{R} \cdot (\mathbf{P} \times \mathbf{Q}) = 0\,, \tag{10}$$

but it's best to learn to do without that affect because the standard literature does not use it. Hereafter, in this first paper, I'll use it at times.

[12] We can also take these cross products in their opposite orders, but introducing a minus sign won't matter, since we are trying to get an inner product of zero, anyway. And minus zero is the same as plus zero.

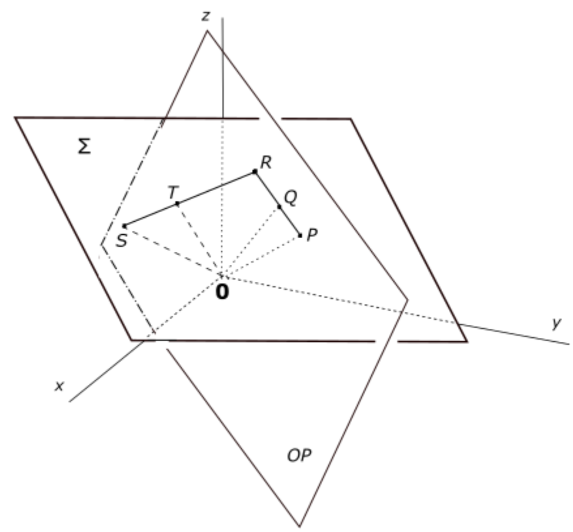

Figure 4. Compared to Figure 3, we've added the points S and T in the projective plane Σ, which lie in a different oriplane than OP, which we'll call OP' (not shown to reduce clutter). Lines ST and PQ will meet at point R.

Now, how do we find the point of intersection of two distinct lines in the projective plane Σ? Refer now to Figure 4. Let R be the point of intersection of line $\overline{PQ}$ and $\overline{ST}$. But how do we represent point R algebraically in terms of the given information of points P, Q, S, and T? Well, this being projective geometry, not Euclidean geometry, we are only concerned with determining the direction of R in $\mathbb{R}^3$. For that limited purpose, all we need is any vector on the line $\overline{\mathbf{0}R}$.

The key to finding the algebraic expression that will give us a representation of the direction of the point R is first to realize that the line $\overline{\mathbf{0}R}$ is along the line of intersection of the two oriplanes OP and OP', and second, to realize that we can find that direction from taking the cross product of the normals to those two oriplanes. (The line of intersection of two planes is orthogonal to the normals of both planes, hence, the cross product of those normals.) Thus, the 'projective location' of the point R is given by the meet of the two lines

$$R \sim (P \times Q) \times (S \times T)\,. \tag{11}$$

If we were to write this equation in terms of 3-space vectors, we would get

$$\mathbf{R} = \lambda(\mathbf{P} \times \mathbf{Q}) \times (\mathbf{S} \times \mathbf{T})\,, \tag{12}$$

where λ is some unknown nonzero real number. However, we don't need to know the value of λ to answer the kind of questions we will encounter in incidence geometry in this series of papers. Hence, we will confidently write the 'projective location'[13] of the point R as given by (effectively setting λ to unity), thus

$$R = (P \times Q) \times (S \times T)\,. \tag{13}$$

One last point to make in this section. There is a unique oriplane in $\mathbb{R}^3$ that is parallel to Σ in the Euclidean sense. Call this plane the *ideal* plane. However, we will not make much use of this plane in this series of articles.

3 Last detours before Pappas's Hexagonal theorem

Before proving Pappas's Hexagonal theorem, we need 1) to add a few definitions for use in the series and to make contact with the literature; 2) to review (briefly) those few aspects of inner products and determinants that we'll use repeatedly in this series to make actual calculations to solve our problems; and 3) to introduce the compact 'bracket' notation. This and the next few sections will accomplish this task. Let's begin.

Definition 1:
Projective Geometry is the area of mathematics concerned with the invariants of figures under projective transformations.

So, what are these invariants of figures under projections? First, what are the figures? The figures, for our restricted purposes, are planar figures of lines and points. The invariants are 1) the indicated points in the given plane, 2) the lines, and 3) the incidence relations between lines and between points on lines. Thus, if two lines are incident in the figure, they are incident in the projection. For our purposes, we consider the projections to be between one plane sitting in 3-space to another plane in 3-space, or a sequence of such projections.

Definition 2:
A collineation is a bijection from one projective space to another that preserves incidences.

Definition 3:
Two distinct lines *meet* (symbolically $\vee$) at a point, and two distinct points are joined by (that is, lie on) a line (symbolically $\wedge$). Thus, if ℓ_1 and ℓ_2 are two lines that meet at point p, we can write: $p = \ell_1 \vee \ell_2$. And if p_1 and p_2 are two points that lie on line ℓ, we can write: $\ell = p_1 \wedge p_2$. Clearly, once ℓ is determined, any two distinct points on that line will have meet equal to ℓ.

[13] I coined this term.

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Projective geometry comes in two flavors: Synthetic and Analytic. In the synthetic version, the subject begins with a statement of the axioms the geometry obeys, and it leave the terms *point* and *line* as undefined (primitive) terms. It also doesn't use coordinates, but, rather, proceeds similar to how synthetic Euclidean geometry is founded on the existence of points and lines and incidences, etc.

Analytic Projective geometry begins typically with $\mathbb{F}^3$, where $\mathbb{F}$ is a field. Our field of choice in this paper is the real numbers.

Besides a knowledge of some linear algebra and Gibbs's vector algebra, the rest of the mathematical preparation the reader will need to follow the proof of Pappus's hexagonal theorem is begun in Chapter 4.

4 Homogeneous Coordinates

First, a little history. The ancient Greeks knew that the conic figures can be obtained by intersecting a plane with a cone. Later, with the idea of projections came the realization that distances and angles are generally not preserved. What is preserved, then? Incidence relationships are. Any two distinct points in a plane determine a line. But the relationship between lines is not so perfect: Any two nonparallel lines in a plane meet at a single point. However, in Euclidean geometry, parallel lines never meet. This seems like a sort of spontaneously broken symmetry between points and lines in a plane.

If we could fix this defect somehow, then there would be a perfect symmetry between theorems about incidence relationships about points and lines in a plane. In other words, to force a duality on the points and lines in a plane so that all theorems about them remain true when we interchange the words 'point' and 'line' is to adopt the convention that all parallel lines in a given direction meet at a single point infinity.

Now imagine having a plane in R^3 with z value unity. That is, the plane $z = 1$. Every line in that plane can be expressed as the set of points of intersection of the $z = 1$ plane with some plane through the origin $(0, 0, 0)$. The general equation that describes such a plane (i.e., an oriplane) is

$$a_1 x + a_2 y + a_3 z = 0\,, \tag{14}$$

where a_1, a_2, a_3 are real numbers. This equation is said to be *homogeneous* because when all its constant terms are collected into one lumped term, that term is zero, which is true if and only if the plane contains the origin.

At this point in the development, we change gears completely by adopting the methods and notations of Gibbs's vector algebra. (Since this paper assumes the reader is already familiar with this algebra, we won't go into a deep presentation of it here.) In this algebra, a vector is a directed line segment from a base point to its tip. The angle between any two vectors can be established by calculating the dot or inner product between the vectors.

Say we have two free nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in 3-space, and we want to find the angle between them. How do we do this? We bring their base points together (at any convenient point in space, since it doesn't matter) and then compute their dot product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta\,, \tag{15}$$

where θ is the angle between the two vectors. When this angle is $\pi/2$, the cosine of the angle is zero and (15) becomes

$$\mathbf{a} \cdot \mathbf{b} = 0\,, \tag{16}$$

in which case $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other.

We can express Equation (14) in vector form this way

$$\mathbf{a} \cdot \mathbf{x} = 0\,, \tag{17}$$

where $\mathbf{x}$ is a vector (with base point at the origin) perpendicular to the vector $\mathbf{a}$, which also has its base point at the origin. The locus of all points $\mathbf{x}$ in $\mathbb{R}^3$ defined by (17) is the oriplane perpendicular to the vector $\mathbf{a}$.[14]

The whole point of homogeneous coordinates is to enforce a one-to-one correspondence between the points on the projective plane and the vectors that start at the origin and end at the points on the plane. This is why I refer to points as vectors and vice versa.

There are two ways to proceed at this point. One way is to explicitly adopt coordinates to prove theorems in projective geometry by choosing ones easy for computations, such as $(0, 0, 1)$ or $(0, 1, 0)$, etc. We lose nothing in generality by choosing such coordinates because all Pappus figures are projectively equivalent.[15] And now it is obvious why we choose the projective plane to have z coordinate unity.

The other way to proceed is to leave coordinates as implicit and just use the power of dot and cross products on arbitrary points in the projective plane, as is the procedure in this paper. On the other hand, Dorwart will use the coordinates for his proofs, whereas I'll use the noncoordinate method. Either way, it's still just the Gibbs's vector algebra adapted for use in projective geometry.

Two questions might occur to the reader at this point. The first is, Why we need the projective plane to have z coordinate unity if we're going to ignore the explicit use of coordinates in the latter way to do projective geometry? The second question is, Why can't we leave the origin in the projective plane, rather than embed the plane into a larger space?

In answer to the first question, If we aren't going to assign particular coordinates to points in the projective plane, there really is no need to use the canonical construction of the plane with z coordinate equal to 1. In this case, all we need is that the projective plane not include the origin, or put another

[14] To ensure that the origin is included in this locus of points, giving us the entire oriplane, we must insist that the zero vector is orthogonal to every vector.

[15] In this concept I include the great freedom allowed to draw the figure in the first place.

way, that the origin is not in the projective plane. And *that* is why projective geometry adds another spatial dimension to the problem.

But surely there are proofs of Pappus's theorem that only use the 2-D plane and not 3-space, right? Right. I've seen proofs using Euclidean geometry, and I invented a proof using isotropic spinors in the plane. I can easily imagine a proof using geometric algebra in the plane. So why do we embed the Pappus planar figure in 3-space? You might say, We do so, so we can get a better perspective on the problem. And, humor, aside, this is quite literally the case.

One reason to embed the projective plane into a space of higher dimension is to deal more effectively with those nasty 'points at infinity', and I'll get to that issue later. The other reason we do this is to avoid irksome special conditions that we'd have to be watchful for if the origin were inside the plane, and which are never a problem if the origin is outside the plane. And this is a nice segue that leads us to a full presentation of the technical aspects of how to represent *line intersection* and *collinearity* in terms of the Gibbs's vector algebra.

5 Technical Preliminaries

Let's get technical with a little vector algebra and linear algebra now in $\mathbb{R}^3$, as well as a review of the previous sections. In the projective plane every pair of distinct lines intersect, or *meet*, at a unique point. If the lines are parallel, they meet at a point at infinity. Otherwise, they meet at a finite point, just like in Euclidean geometry. Every line in the projective plane is the intersection of a unique plane through the origin and the projective plane. Henceforth, we'll refer to the projective plane as Σ.

Every plane through the origin can be uniquely defined (up to a sign) by its unit normal vector, call it $\mathbf{n}$, where $\mathbf{n} \cdot \mathbf{n} = 1$. The equation that defines the locus of points $\mathbf{x}$ in space that are in the plane perpendicular to $\mathbf{n}$ is given by

$$\mathbf{n} \cdot \mathbf{x} = 0\,. \tag{18}$$

However, because (18) is homogeneous, we can be more general in our characterization of this plane by an arbitrary nonzero rescaling of $\mathbf{n}$. So, let λ be any nonzero real number, and let $\mathbf{a} = \lambda\mathbf{n}$, then by multiplying (18) through by λ, the equation can be equivalently recast as

$$\mathbf{a} \cdot \mathbf{x} = 0\,. \tag{19}$$

Now, so long as this vector $\mathbf{a}$ is not normal to Σ, the plane normal to $\mathbf{a}$ will intersect Σ in the Euclidean sense. So, say there are two distinct lines in Σ, L_1 and L_2, say, that intersect at point p. Then there are two distinct plane Σ_1 and Σ_2 through the origin that intersect Σ in lines L_1 and L_2, respectively. (Consult Figure 5.)

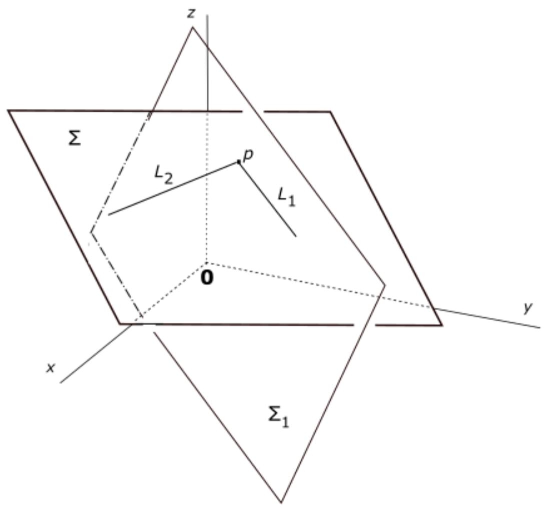

Figure 5. Line L_1 in oriplane Σ_1 meets line L_2 in oriplane Σ_2 (not shown to reduce clutter) at point p.

Let Σ_1, Σ_2, and Σ, be given by, respectively,

$$a_1x + a_2y + a_3z = 0\,, \tag{20a}$$

$$b_1x + b_2y + b_3z = 0\,, \tag{20b}$$

$$c_1x + c_2y + c_3z = d\,, \tag{20c}$$

where d and all the coefficients are real numbers, and in particular, d is nonzero. Or, expressed in matrix form

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}\,. \tag{21}$$

The simultaneous solution to these equations gives us the exact point $\hat{p}$ in Σ, namely,

$$\hat{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \frac{d}{\det M} \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}\,, \tag{22}$$

where M is the coefficient matrix in (21), and its determinant is nonzero:

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\,. \tag{23}$$

Now, we don't need the exact point $\hat{p}$ in Σ. We only need the direction of $\hat{p}$ to characterize it for our purposes. Any vector along the ray defined by $\hat{p}$ will do.

To that end, we set

$$\mathbf{p} = \begin{bmatrix} b_1c_2 - b_2c_1 \\ a_2c_1 - a_1c_2 \\ a_1b_2 - a_2b_1 \end{bmatrix}, \tag{24}$$

where we have ignored nonzero factors of this vector that appeared in (22).

The reader might suspect that, just as the vector[16] $\mathbf{a} = [a_1, a_2, a_3]^{\mathrm{T}}$ is perpendicular to Σ_1 and vector $\mathbf{b} = [b_1, b_2, b_3]^{\mathrm{T}}$ is perpendicular to Σ_2, that vector vector $\mathbf{c} = [c_1, c_2, c_3]^{\mathrm{T}}$ is perpendicular to Σ, and this is correct. Let's prove it.

Let $\mathbf{X} = [X_1, X_2, X_3]^{\mathrm{T}}$ be some particular point that satisfies (20c), that is, is a point on Σ, and let $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ be any other point on Σ. Now, since $\mathbf{X}$ and $\mathbf{x}$ are both points in Σ then their difference is a vector lying in the plane Σ. So, substitute $\mathbf{X}$ and $\mathbf{x}$ into (20c) and subtract the resulting two equations to get

$$c_1(X_1 - x_1) + c_2(Y_2 - y_2) + c_3(X_3 - x_3) = 0\,. \tag{25}$$

But this equation can be rewritten as

$$\mathbf{c} \cdot (\mathbf{X} - \mathbf{x}) = 0\,. \tag{26}$$

But since $\mathbf{X} - \mathbf{x}$ is an arbitrary vector in Σ and $\mathbf{c}$ is normal to it, then $\mathbf{c}$ is normal to Σ.

A useful feature of the determinant of a square matrix is that the effect of swapping any two rows of the matrix will only multiply the resulting determinant of the new matrix by -1. Therefore, any sequential even number of such swaps with leave the value of the resulting determinant unchanged, and any odd number will multiply it by -1. A cyclic permutation of the rows of a 3×3 matrix is an even number of swaps and so will not change the sign of the determinant.

Let's cyclically permute the rows of M in (23), bringing the bottom row to the top:

$$M' = \begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}, \tag{27}$$

and the determinant of M' is

$$\det M' = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{28}$$

Normally, the determinant is scalar-valued, but Gibbs's algebra allows us to write vector-valued determinants, like this one

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\,, \end{aligned} \tag{29}$$

[16]The superscript T means to take the matrix transpose.

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the directions of the x, y, z-axes, respectively. Of course, $\mathbf{a} \times \mathbf{b}$ is the cross product of $\mathbf{a}$ and $\mathbf{b}$, and is a vector orthogonal to both of these vectors. Now, if we dot this through by some arbitrary vector $\mathbf{c}$, we get

$$\begin{aligned}\mathbf{c} \cdot \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3 \,. \end{aligned} \tag{30}$$

To resolve any ambiguity here, the expression $\mathbf{c} \cdot \mathbf{a} \times \mathbf{b}$ is always to be interpreted as $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.

By straightforward calculations using (30), one can prove that

$$\mathbf{a} \cdot \mathbf{a} \times \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \times \mathbf{b} = 0 \,. \tag{31}$$

This last result is also known from linear algebra, which proves that the determinant of any nontrivial $n \times n$ matrix is zero if any two rows have the same components.

Lemma:

$$\mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \,. \tag{32}$$

We now have an interpretation of (28) in terms of the dot and cross products of vectors built out of the rows of the matrix M'. We can generalize our claim from linear algebra about when the determinant of a square matrix is zero. A determinant is zero if and only if its rows are linearly dependent.[17] A vector is linearly dependent on a set of other vectors, if it can be written as a linear combination of the other vectors.

In the case of the three vectors, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, say that $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$. We'll view this linear dependence both algebraically and geometrically. First, algebraically. Then we can write for some nonzero scalars α and β

$$\mathbf{c} = \alpha \mathbf{b} + \beta \mathbf{c} \,. \tag{33}$$

Then, applying (31),

$$\mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = (\alpha \mathbf{a} + \beta \mathbf{b}) \cdot \mathbf{a} \times \mathbf{b} = \alpha \mathbf{a} \cdot \mathbf{a} \times \mathbf{b} + \beta \mathbf{b} \cdot \mathbf{a} \times \mathbf{b} = 0 \,. \tag{34}$$

Viewed geometrically, this means that $\mathbf{c}$ is in the same plane as determined by $\mathbf{a}$ and $\mathbf{b}$. Then, since $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane containing $\mathbf{a}$ and $\mathbf{b}$, it is orthogonal to $\mathbf{c}$ as well. That is, $\mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = 0$.

[17] We won't have to concern ourselves with the case when one of the rows is the zero vector, because the zero vector (i.e., the origin) is not a point of the projective plane—by design.

6 Line Intersection and Collinearity

Let $\mathbf{a}$ and $\mathbf{b}$ be distinct points in Σ. Then the plane through the origin that contains these points intersects Σ in a line that contains points $\mathbf{a}$ and $\mathbf{b}$. A vector normal to this plane is the vector $\mathbf{a} \times \mathbf{b}$. Thus the join of points $\mathbf{a}$ and $\mathbf{b}$ in Σ, that is, the line ℓ containing them, is characterized briefly by $\mathbf{a} \times \mathbf{b}$. The set of points on ℓ is the locus of points $\mathbf{x} \neq 0$ in Σ orthogonal to $\mathbf{a} \times \mathbf{b}$. In other words, $\mathbf{x}$ is 'on the line' ℓ if and only if

$$\mathbf{x} \cdot \mathbf{a} \times \mathbf{b} = 0\,, \tag{35}$$

in which case we say that points $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{b}$ are *collinear* in the projective sense.

"Now wait a minute!" I hear you say. "Didn't you claim recently that an entire oriplane orthogonal to the vector $\mathbf{a} \times \mathbf{b}$ is defined by Equation (35), not just a point on a line?" To which I reply that that's right, when regarding (35) as an equation in $\mathbb{R}^3$. But now consider a ray from the origin to infinity in that oriplane. It must intersect the line ℓ in some point. However, considered projectively, every point on that ray is equivalent to that point of intersection because they all share one critical distinction in $\mathbb{R}^3$, namely, they all share the same direction in 3-space.

Definition: The *triple scalar product* of the three arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ shall be given by $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$.

Now for the join of two lines in the projective plane. We've already seen that a line in Σ is defined by two distinct points in Σ, and since the meet of two lines in the projective plane is a point, we need four distinct points, two for each line, to define this meet of the lines. Let $\mathbf{x}$ be this unique point of intersection of these two lines, $\ell_1 = \mathbf{a} \times \mathbf{b}$ and $\ell_2 = \mathbf{c} \times \mathbf{d}$. Since $\mathbf{x}$ lies on each line, it must satisfy the two equations

$$\mathbf{x} \cdot \mathbf{a} \times \mathbf{b} = 0 \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{c} \times \mathbf{d} = 0\,. \tag{36}$$

There is an obvious choice for the solution to this couple of constraints, and that is

$$\mathbf{x} = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})\,. \tag{37}$$

This solution for $\mathbf{x}$ is unique, up to arbitrary nonzero scale factor, because two distinct lines can only intersect on a single point. And as a reminder, $\mathbf{x}$ as computed by Equation (37) is said to be the 'projective location' of the point $\mathbf{x}$.

However, the triple cross product is not too convenient for calculations for my tastes. Let's employ the double cross product vector identity to reexpress it more simply:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\mathbf{A} \cdot \mathbf{C} - \mathbf{C}\mathbf{A} \cdot \mathbf{B}\,, \tag{38}$$

a result familiar to most calculus students. So, start with

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = -[(\mathbf{c} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{b})]\,, \tag{39}$$

and then set

$$\mathbf{A} \to \mathbf{c} \times \mathbf{d}, \quad \mathbf{B} \to \mathbf{a}, \quad \mathbf{C} \to \mathbf{b}\,, \tag{40}$$

and then, using (38), (39) continues on with

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= -[(\mathbf{c} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{b})] \\ &= -[\mathbf{a}(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b} - \mathbf{b}(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}] \\ &= \mathbf{b}(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a} - \mathbf{a}(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b} \\ &= \mathbf{ba} \cdot (\mathbf{c} \times \mathbf{d}) - \mathbf{ab} \cdot (\mathbf{c} \times \mathbf{d})\,. \qquad (41\text{a})\end{aligned}$$

However, vectors **a** and **b** don't get all the glory. By a similar maneuver, we can write

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{ca} \cdot (\mathbf{b} \times \mathbf{d}) - \mathbf{da} \cdot (\mathbf{b} \times \mathbf{c})\,. \tag{41b}$$

Hint:

$$\mathbf{A} \to \mathbf{a} \times \mathbf{b}, \quad \mathbf{B} \to \mathbf{c}, \quad \mathbf{C} \to \mathbf{d}\,, \tag{42}$$

and then use cyclic permuting of the vectors in the triple scalar products.

7 The Compact Bracket Notation

Convention in this subject employs a 'bracket' notation to make the equations in projective geometry much easier write down and to comprehend. I wholeheartedly endorse this practice. Let's look at my take on them.

Look at Equations (41a) and (41b). They're just vector equations in the Gibbs's algebra. But if we include every dot and cross product, the expressions and equations would be full of distracting symbols that really aren't needed. The proof that they're not needed is that we can define a bracket notation that unambiguously does away with them.

So how does this bracket notation work? So far, I have found only four bracket types to use in proving theorems in projective geometry. They are

$$[\mathbf{a}], \quad [\mathbf{ab}], \quad [\mathbf{abc}], \quad [\mathbf{abcd}]\,, \tag{43}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are vectors/points in the projective plane. As is true in the Gibbs's vector algebra, all these expressions represent either scalars or vectors, though vectors have two different interpretations in projective geometry: The following three expressions

$$[\mathbf{a}], \quad [\mathbf{ab}], \quad [\mathbf{abcd}]\,, \tag{44}$$

are vector-valued, as given by the following definitions:

$$[\mathbf{a}] \equiv \text{the point } \mathbf{a} \text{ in the projective plane}, \tag{45a}$$

$$[\mathbf{ab}] \equiv \mathbf{a} \times \mathbf{b} \text{ represents a line in the projective plane}, \tag{45b}$$

$$[\mathbf{abcd}] \equiv (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) \text{ represents a point in the projective plane.} \tag{45c}$$

In terms of the jargon already laid out, [**ab**] is the join of points **a** and **b** (i.e., the line containing **a** and **b**). And [**abcd**] is the meet of lines [**ab**] and [**cd**]. Whereas,

$$[\mathbf{abc}] \equiv \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}\,, \tag{46}$$

which, as a reminder, is the *triple scalar product*. We've already seen this quantity to represent a determinant, with the properties

$$[\mathbf{abc}] = [\mathbf{cab}] = [\mathbf{bca}] = -[\mathbf{bac}] = \text{ etc}\,. \tag{47}$$

Of course, if any one of the triple scalar products in (47) is zero, they all are. And the scalar product of three nonzero vectors is zero if and only if they're coplanar.

One final obvious point to make is that, based on (31), for all $\mathbf{a}, \mathbf{b}$

$$[\mathbf{aab}] = [\mathbf{bab}] = 0\,. \tag{48}$$

In the Pappus theorem, points in the figure are labeled with subscripts. I could represent the point A_1, for example, as $[A_1]$, but, in the interest of minimality, I should go all the way and just write [1] for it. And this is allowable so long as it doesn't introduce an ambiguity.

I can also mix subscripted with unsubscripted points in a bracket. For example, to claim that the three points A_2, u, and A_3 are collinear, all I need write is[18]

$$[2u3] = 0\,, \tag{49}$$

which, of course, stands for

$$A_2 \cdot u \times A_3 = 0\,. \tag{50}$$

Equation (38) for the double cross product can be recast as

$$\begin{aligned}[\mathbf{A}] \times [\mathbf{BC}] &= [\mathbf{B}]\mathbf{A} \cdot \mathbf{C} - [\mathbf{C}]\mathbf{A} \cdot \mathbf{B} \\ &= [\mathbf{B}][\mathbf{A}] \cdot [\mathbf{C}] - [\mathbf{C}][\mathbf{A}] \cdot [\mathbf{B}]\,.\end{aligned} \tag{51}$$

Lastly, we look at some identities involving the triple cross product $[abcd]$. Remember that this is the meet point of the two lines $[ab]$ and $[cd]$. In our bracket notation, (39) becomes

$$[abcd] = -[cdab]\,. \tag{52}$$

Note that $[abcd]$ also changes sign if we transpose either a and b, or c and d. And (41a) and (41b) become

$$\begin{aligned}[abcd] &= [b][acd] - [a][bcd] \\ &= [c][abd] - [d][abc]\,.\end{aligned} \tag{53}$$

[18] For the time being, I will treat boldface and non-boldface vectors the same. Perhaps later I'll find a reason to distinguish them.

EXERCISE: An Identity

Start with $[abcd] = [b][acd] - [a][bcd]$ and take the cross product of it on both sides of the equation on the left with vector $e = [e]$, to get[19]

$$[ab][ecd] - [cd][eab] = [eb][acd] - [ea][bcd]\,. \tag{54a}$$

Now, dot (54a) with vector $c = [c]$ on both sides to get

$$[cab][ecd] = [ceb][acd] - [cea][bcd]\,. \tag{54b}$$

Lemma 1 (on the meet of distinct lines)
Let A, B, C, D be distinct points in the projective plane. Then

$$[ABCD] = B[ACD] - A[BCD] \tag{55a}$$
$$= C[ABD] - D[ABC]\,. \tag{55b}$$

Remember that $[ABCD]$ is a vector/point and is equal to $(A \times B) \times (C \times D)$

Lemma 2 (on the collinearity of points)
Three distinct points A, B, C in the projective plane are collinear if and only if $[ABC] = 0$.

More Practice

Let's do some examples scalarizing vectors. Say we have the vector equation

$$A = \lambda B + C\,, \tag{56a}$$

where A, B, C are vectors and λ is a scalar. If we cross (56a) by, say, the vector D on the right, we get

$$A \times D = \lambda B \times D + C \times D\,. \tag{56b}$$

Now we can dot this last equation on the left by, say, the vector E to get

$$E \cdot A \times D = \lambda E \cdot B \times D + E \cdot C \times D\,, \tag{56c}$$

where, again, the expression $A \cdot B \times C$ is always to be interpreted as $A \cdot (B \times C)$. And, of course, we want to rewrite the last three equations into our bracket notation, yielding

$$[A] = \lambda[B] + [C]\,, \tag{57a}$$

$$[AD] = \lambda[BD] + [CD]\,, \tag{57b}$$

$$[EAD] = \lambda[EBD] + [ECD]\,. \tag{57c}$$

[19]Hint: Remember that $[abcd]$ is a cross product $[ab] \times [cd]$, so that $[e] \times [abcd] = [e] \times ([ab] \times [cd])$, so use the double cross product formula (51).

To demonstrate dotting a vector equation with a cross product, consider dotting (56a) with $D \times E$ to get

$$A \cdot D \times E = \lambda B \cdot D \times E + C \cdot D \times E\,, \tag{58a}$$

or, in bracket notation,

$$[ADE] = \lambda[BDE] + [CDE]\,. \tag{58b}$$

A shorthand way to think about how we went from (57a) to (57c), was to apply the operation of $[E\llcorner D]$ across (57a). And the alternative way to scalarize a vector equation is to apply the following cross product $[\llcorner DE]$, such as in going from (57a) to (58b).

8 Theorem of Pappus

So far as I can tell, Pappus merely stated the following theorem, but never proved it:

As shown in Figure 6, let there be two distinct lines in a plane, each having three distinct points on them, A_1, A_3, A_5 on one, and A_2, A_4, A_6 on the other. (A particular labeling of these points is quite arbitrary.) Then, the points of the intersections (i.e., the meets) of the given pairs of interior lines (i.e., joins) are collinear. That is, points u, v, w are on the same line.

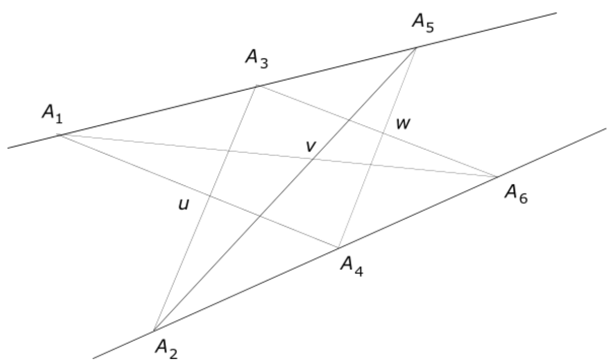

Figure 6. One version of Pappus's hexagonal planar figure.

Proof of Pappus's Theorem: STEP 1 TO THE PROOF: Collect the constraints.

As presented in the figure, $u = \overline{A_1A_4} \vee \overline{A_2A_3}$, $v = \overline{A_1A_6} \vee \overline{A_5A_2}$, and $w = \overline{A_3A_6} \vee \overline{A_5A_4}$, or, presented in our compact notation

$$u = [1423]\,, \tag{59a}$$
$$v = [1652]\,, \tag{59b}$$
$$w = [3654]\,. \tag{59c}$$

Thus, we can recast the essence of the claim of the theorem in this form: Show that

$$[uvw] = 0\,, \tag{60}$$

given that 1)

$$[135] = 0\,, \tag{61a}$$

$$[246] = 0\,, \tag{61b}$$

and given that 2)

$$[1u4] = 0\,, \tag{62a}$$

$$[3u2] = 0\,, \tag{62b}$$

$$[1v6] = 0\,, \tag{62c}$$

$$[5v2] = 0\,, \tag{62d}$$

$$[3w6] = 0\,, \tag{62e}$$

$$[5w4] = 0\,. \tag{62f}$$

STEP 2 TO THE PROOF: In order to show that $[uvw] = 0$, find expressions for u, v, w in terms of the given points.[20]

Thus, using (55a), we can solve for $A_1 = [1]$, $A_3 = [3]$, $A_5 = [5]$, respectively, by solving for their projective locations, as follows:

$$[1] = [u4v6] = [4][uv6] - [u][4v6]\,, \tag{63a}$$

$$[3] = [u2w6] = [w][u26] - [6][u2w]\,, \tag{63b}$$

$$[5] = [v2w4] = [2][vw4] - [v][2w4]\,. \tag{63c}$$

These can be rewritten as

$$[u][4v6] = [4][uv6] - [1]\,, \tag{64a}$$

$$[w][u26] = [6][u2w] + [3]\,, \tag{64b}$$

$$[v][2w4] = [2][vw4] - [5]\,. \tag{64c}$$

Now we take the scalar product[21] of the last three equations to get

$$\alpha[uvw] = ([4][uv6] - [1]) \cdot \{([2][vw4] - [5]) \times ([6][u2w] + [3])\}\,. \tag{65}$$

where $\alpha = [4v6][2w4][u26]$. Now, $\alpha \neq 0$ because none of its three factors is zero, according to Lemma 2. To expand the RHS[22] of this, we first distribute the cross products. In doing so, we lose the term containing [426] because of (61b), and the term containing [153] because of (61a).

[20]There are many ways to go about doing this. The way I chose to do this might not be the best way, but it worked.

[21]In this paper, *scalar product* means, when applied to **three** vectors, a, b, c, is $[abc] = [a] \cdot ([b] \times [c])$.

[22]RHS means 'right-hand side.'

Expanding, we get

$$[uvw] \doteq [423][uv6][vw4] - [456][uv6][u2w] - [453][uv6]$$
$$- [126][vw4][u2w] - [123][vw4] + [156][u2w]\,, \tag{66}$$

where the overdot on the equal sign means that LHS of (66) is equal to the RHS up to the nonzero scalar factor α. Ordinarily, such a condition is a truism, but it works here since all we want to show is that is the RHS is zero, then so is also the LHS, so that the nonzero factor α is irrelevant.

STEP 3 TO THE PROOF: Conform the vector constraints to use in the scalar equation (66) and simplify in successive steps.

We observe that the terms in (66) are scalars, but the constraints we must invoke to finish this proof are vectors. Thus, to consistacize them, the simpler thing to do is to scalarize the vector constraints by either 1) crossing them with one vector and then dotting that result with another, or else 2) by dotting them with a cross product (that is taking their 'scalar product'). In either case we end up with a scalar in the form: $x \cdot y \times z$.

Now, I want to scalarize equations (64a)–(64c) in such a way so as to employ as many of the constraints found in (62a)–(62f) as I can, because it's by these constraints that the Pappus figure is properly defined.[23] Applying $[2\textvisiblespace 3]$ to (64a), we get

$$[2u3][4v6] = [243][uv6] - [213]\,, \tag{67}$$

and then applying (62b), we have[24]

$$[243][uv6] - [213] = 0\,. \tag{68a}$$

Similarly, applying $[1\textvisiblespace 6]$ to (64c) and then applying (62c), we get

$$[126][vw4] - [156] = 0\,. \tag{68b}$$

Lastly, applying $[4\textvisiblespace 5]$ to (64b) and then applying (62f), we get

$$[465][u2w] + [435] = 0\,. \tag{68c}$$

Returning to (66), we see that, on factoring $[u2w]$ out of the fourth and sixth terms on the RHS, the two terms cancel each other because of (68b), leaving us with

$$[uvw] \doteq [423][uv6][vw4] - [456][uv6][u2w] - [453][uv6] - [123][vw4]\,. \tag{69}$$

By a similar process we see that the first and fourth terms cancel because of (68a), leaving us with, after factoring and using (68c),

$$[uvw] \doteq -([456][u2w] + [453])[uv6] = 0\,. \tag{70}$$

[23] In other words, it would be very strange to be able prove a fundamental result concerning the Pappus figure without employing the information contained in the constraint equations that define that figure.

[24] Since [3u2]=0 and [2u3]=-[3u2] then [2u3]=0.

Thus

$$[uvw] = 0\,, \tag{71}$$

as we needed to show. ■

9 Conclusion

Real projective geometry of the plane looks complicated, but really it isn't that bad. All we're doing is to use the given information, converted to equations in bracket notation, to prove the theorem statement, which is also converted into an equation in bracket notation. True enough, though, these theorem proofs can generate a lot of steps.

This paper proves that the Gibbs's vector algebra can be co-opted to use as a means of finding joins and meets in the projective plane: the algebra is exactly the same as if the Gibbs's operation were being done on ordinary vectors in Cartesian geometry of $\mathbb{R}^3$; only the meaning of the operations has changed.

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