

Stereographic Projection of a Circle

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Abstract

This paper proves a simple result for the stereographic projection of a circle onto a line, as a prior step to proving the theorem for the projection of a sphere to a plane.

1 Introduction

We begin with the projection of a circle onto a line (in this case the x -axis) by a line from the North Pole of the circle to the x -axis at point $\mathbf{x} = x\mathbf{e}_1$, intercepting the circle at point $\mathbf{x}' = x'\mathbf{e}_1 + y'\mathbf{e}_2$. Our task is to solve for the components of \mathbf{x}' in terms of the value x . See Figure 1.

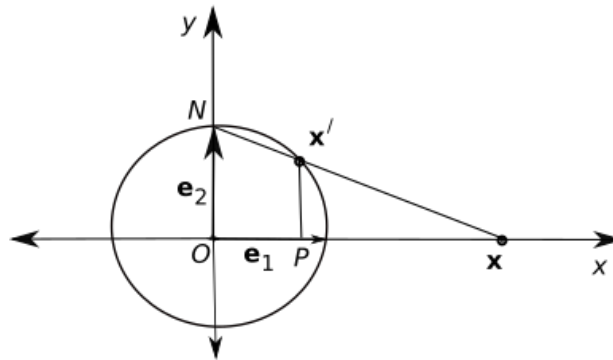


Figure 1. Depicted is a circle of radius unity, centered at the origin of \mathbb{R}^2 . The line from the North Pole of the circle meets the x -axis at point $\mathbf{x} = x\mathbf{e}_1$ and intercepts the circle at point $\mathbf{x}' = x'\mathbf{e}_1 + y'\mathbf{e}_2$. Point P results from dropping a perpendicular from \mathbf{x}' to the x -axis.

2 Proof

Our proof is straightforward. We begin with the basic equation for \mathbf{x}' :

$$\mathbf{x}' = x' \mathbf{e}_1 + y' \mathbf{e}_2, \quad (1)$$

which has length constraint

$$1 = x'^2 + y'^2. \quad (2)$$

To me, the simplest way to proceed is to use similar triangles.¹

$$\Delta N\mathbf{x}O \sim \Delta \mathbf{x}'\mathbf{x}P. \quad (3)$$

From this we get

$$\frac{x}{1} = \frac{x - x'}{y'}. \quad (4)$$

On squaring both sides we get

$$x^2 y'^2 = (x - x')^2. \quad (5)$$

On eliminating y' between this last equation and (2), we get

$$x^2(1 - x'^2) = x^2 - 2xx' + x'^2. \quad (6)$$

Solving this for x' , we get

$$x' = \frac{2x}{1 + x^2}. \quad (7)$$

We obtain y' from (2):

$$y'^2 = 1 - x'^2 = 1 - \frac{4x^2}{(1 + x^2)^2} = \frac{(1 + x^2)^2 - 4x^2}{(1 + x^2)^2} = \frac{(1 - x^2)^2}{(1 + x^2)^2}. \quad (8)$$

Hence,

$$y' = \pm \frac{1 - x^2}{1 + x^2}. \quad (9)$$

Choosing the sign correctly gives us

$$\mathbf{x}' = \frac{2x}{1 + x^2} \mathbf{e}_1 - \frac{1 - x^2}{1 + x^2} \mathbf{e}_2. \quad (10)$$

¹By 'simplest' I mean 'least sophisticated'. A more sophisticated way to proceed is to set up the equation $\mathbf{x}' = \mathbf{x} + \lambda(\mathbf{e}_2 - \mathbf{x})$, where $\mathbf{x} = x\mathbf{e}_1$. Then, solve for λ by using the given constraints. If you already know about vectors, then this vector approach has its definite merits.