The Angle Bisector Theorem (of a Triangle)

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Abstract

This paper uses old-fashioned geometry to prove the Angle Bisector Theorem, but does so in a (possibly) novel way.

1 Introduction

Definition: Two triangles are are said to be *similar* if their corresponding angles are congruent (have the same measures).

Lemma: The corresponding sides of similar triangles are in the same proportion.

The Angle Bisector Theorem is ancient and has many proofs. Let's consider a (possibly) new proof here. Refer to the triangle ABC in Fig. 1.

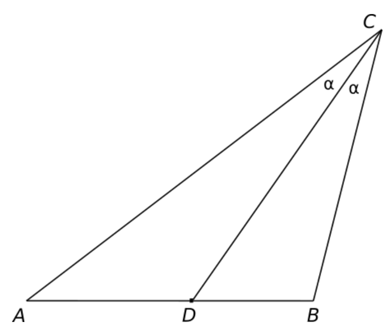

Figure 1. The triangle defined by the vertices A, B, C, has a point D placed on line segment $\overline{AB}$ so as to bisect the angle at vertex C.

The triangle defined by points A, B, C is an arbitrary triangle. The line segment $\overline{CD}$ bisects the angle at vertex C. Then, the point D divides the line segment $\overline{AB}$ as per the following proportion: $\dfrac{AC}{AD} = \dfrac{BC}{BD}$.

2 Step One

Construct a circle of radius CB and centered at C, as in Fig. 2. The point F is the meet of the extension of line segment $\overline{AC}$ to the circle. Point E is the meet of the circle with line segment $\overline{AC}$.

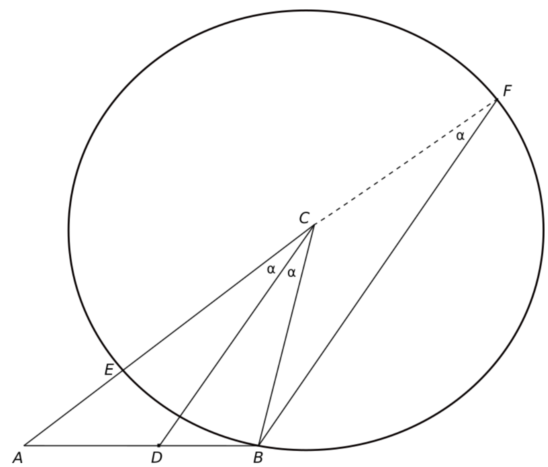

Figure 2. After constructing the circle centered at C of radius CB, we use the Inscribed Angle Theorem to show that the angle at F is half the central angle of ECB (i.e., 2α), thus is equal to α.

3 Step Two

There is a theorem from geometry that if two lines cut a third line at equal angles, the two lines are parallel. From the definition above, since the corresponding angles of the two triangles FAB and CAD are congruent (the reader may prove this), the triangles are similar. Thus, by the Lemma above, the corresponding parts of the two triangles are proportional, therefore,

$$\frac{AC}{AD} = \frac{AF}{AB} = \frac{AC + CF}{AB} = \frac{AC + CB}{AB} = \frac{AC}{AB} + \frac{CB}{AB}\,. \tag{1}$$

On subtracting $\frac{AC}{AB}$ from both sides, we get,

$$AC\left[\frac{1}{AD} - \frac{1}{AB}\right] = \frac{CB}{AB}\,. \tag{2}$$

Using that $AB - AD = DB$ and applying some algebra, we get that

$$\frac{AC}{AD} = \frac{BC}{BD}\,. \tag{3}$$