Is A Right Triangle Inscribed in a Circle?

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1 Setting Up

Better expressed, When, if ever, is a right triangle inscribed in a circle? More specifically, let the vertex at which is the right angle of a right triangle be called the 'apex' of the triangle, just to give it a convenient label. Let the right triangle have its hypotenuse exactly over a diameter of a circle of radius r. Then, of course, the apex of the triangle is either inside the circle, outside the circle, or on the circle. I intend to prove that it must be **on** the circle.

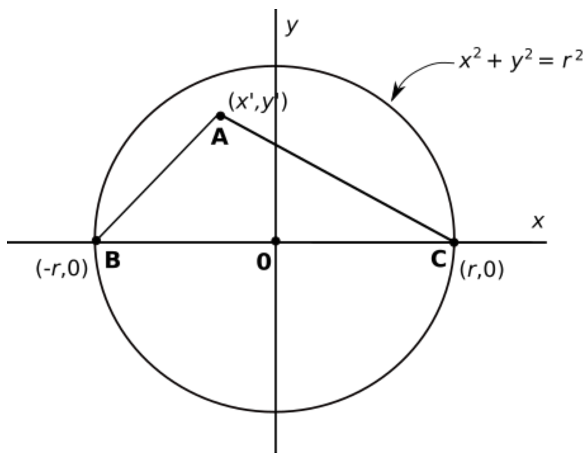

Figure 1. Let points $\mathbf{B}, \mathbf{C}$ be points on the circle, being connected by a diameter of the circle, whose center is at point $\mathbf{0}$. Point $\mathbf{A}$ is the apex point of the triangle. Therefore, as explained above, the non-apex points of the triangle are on points $\mathbf{B}, \mathbf{C}$, and the angle at $\mathbf{A}$ is a right angle.

2 Proof

Since our proof will be by analytic geometry, we will need to assign coordinates to all relevant points in the plane. The Apex point has been given the generic coordinates of (x', y'). So far, all we know about its coordinates is that $y' \neq 0$. The circle has equation

$$x^2 + y^2 = r^2 \,. \tag{1}$$

Thus, to show that the Apex point lies on the circle, its coordinates must satisfy this equation:

$$x'^2 + y'^2 = r^2 \,. \tag{2}$$

Now, since $\triangle\mathbf{ABC}$ is a right triangle, we can relate the lengths of its three sides by the Pythagorean Theorem.

$$(\mathbf{A} - \mathbf{B})^2 + (\mathbf{A} - \mathbf{C})^2 = (\mathbf{B} - \mathbf{C})^2 = 4r^2 \,. \tag{3}$$

In coordinates,

$$(x' - -r)^2 + (y' - 0)^2 + (x' - r)^2 + (y' - 0)^2 = 4r^2 \,, \tag{4}$$

which simplifies down to

$$x'^2 + y'^2 = r^2 \,. \tag{5}$$

Thus, the Apex point lies on the circle.