Euclidean Geometry as a Foundation to the Algebra of Real Numbers

Patrick Reany
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Abstract. Here we construct a crude model of the real numbers out of the axioms of Euclidean Geometry with the help of some intuitive concepts of limit operations and the extraction of roots. Rational numbers are easily constructed using only straight-edge and compass operations. Certain irrational numbers are geometrically demonstrated to exist without the use of the Pythagorean theorem or any special angle. The central element of our constructions is the fact that similar triangles are 'real' multiples of each other, and by which we constantly emphasize the nature of proportions in algebra. Some of the axioms and theorems of algebra are given visual representations.

I. Introduction

The material presented here will be at two levels: The first level requires only a basic knowledge of points, lines, planes, circles, and triangles of Euclidean geometry. The second level requires a knowledge of maps, especially linear maps.

Unfortunately, for this paper all our figures will be static and non-interactive, but much can be accomplished by so-called "paper and pencil" constructions.

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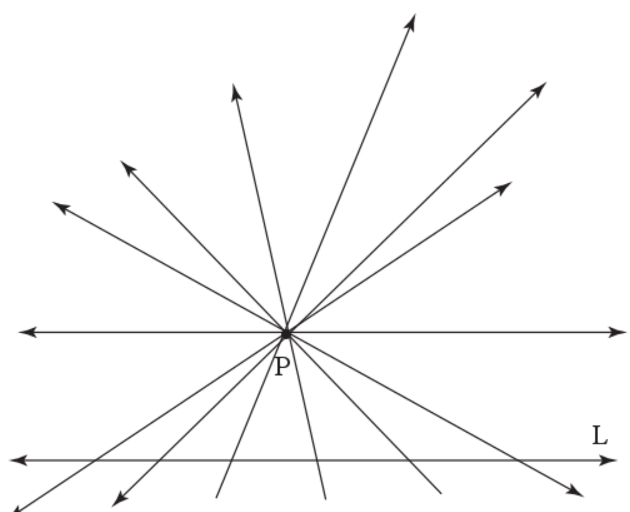

Figure I 1. A line L, a point P not on L, and the pencil of all lines through P.

Consider the figure above. We have in it a line *L*, a point *P* not on *L*, and the pencil of all lines through *P*. Now, except for a single line that goes through P parallel to L, all the rest intersect *L*, and there is a one-to-one correspondence between distinct lines through P intersecting *L* and points of intersection. If you think for a moment about how to use this construction to build a number system upon *L*, it comes to mind that a good place to start would be to choose two points of intersection of the lines through P and the line *L*, labeling one point as ‘0’ and the other as ‘1’.

Although our approach so far looks promising, it fails to give us a precise meaning to the pencil of lines through *P*. However, if we could systematically identify points on the line *L*, we could draw lines through those points and through *P*, and each line would be a member of the pencil of lines through *P*. Thus, we have arrived at our first important observation: That the characterization of the pencil of lines through *P* can be made in terms of a number system built on top of *L*!

Let’s choose our ‘0’ and ‘1’ points and then use our compass to determine the ‘length’ of the line segment between them, and then mark off equal intervals from ‘0’ and ‘1’ in both directions, labeling then with integers as we go. We get the drawing as in Fig. I.2.

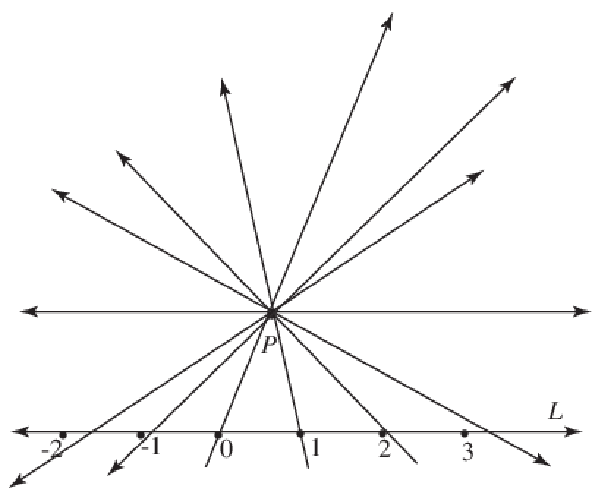

Figure I.2. The integers are partly constructed on *L* using a compass repeatedly marking off equal intervals.

We will avoid deep technical issues and merely assume that the line *L* is the real line, which is really quite adequate for our purposes here.

Thus, we are free to choose any two distinct points on *L* and label them 0 and 1, which then defines a unit distance. What we need now is a notion of the continuity of the Euclidean line, and we must develop this systematically. Intuitively, we think of the Euclidean line as being continuous, meaning that no matter where any line intersects *L*, it intersects it in a point common to both lines. Another way to think about this continuity is to visualize an inclined line moving from left to right, intersecting our newly made number line as it goes (Fig. I. 3.).

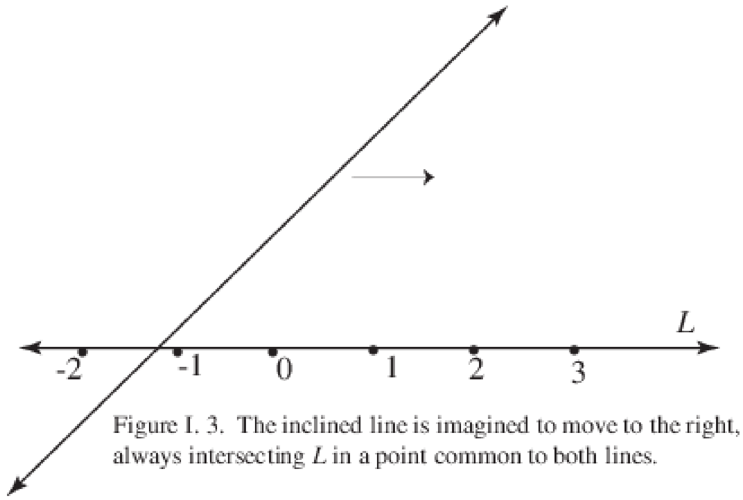

Figure I. 3. The inclined line is imagined to move to the right, always intersecting L in a point common to both lines.

Thus, the Euclidean (real) line is imagined to be without any holes in it. The next issue is how we begin to characterize the points in between the integers. We can start by simply subdividing a given line segment into n congruent subsegments. In Fig. 1.4a we illustrate the problem for $n = 5$, which uses the equally spaced integer points on the line that we have already constructed with ruler and compass operations.

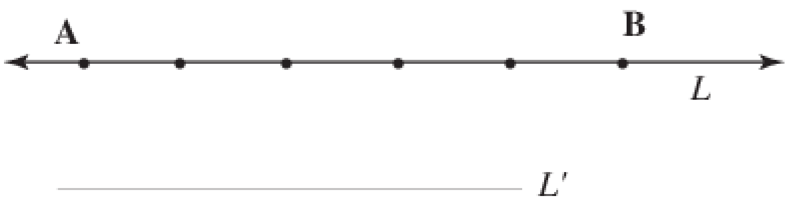

Figure I. 4a. Given a line L with six equally spaced points on it, how can we use it to subdivide an arbitrary line segment L' into five congruent sub line segments?

To finish this problem, we rotate and translate L as is needed to place its leftmost point **A** coincident with the left end of L'. Then we draw a line through **B** and the rightmost point of L'. To finish, we merely draw lines through the remain highlighted points of L parallel to the line drawn through **B**. The resulting construction is depicted in Fig. 1.4b.

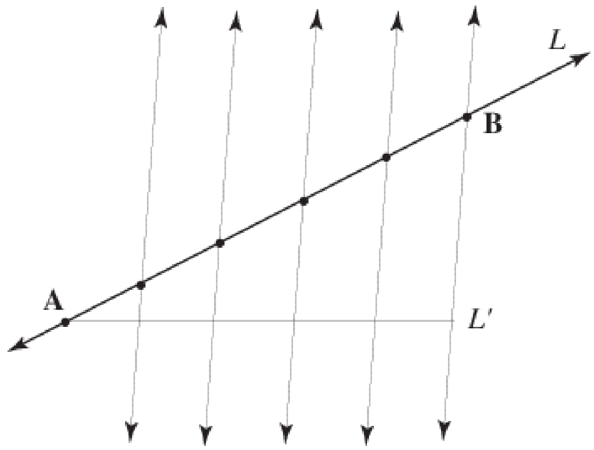

Figure I. 4b. The final state after the line segment L' has been subdivided into five congruent line segments.

To prove that L' has been subdivided into congruent parts requires only an application of theorems on similar triangles. Note that our construction has created a series of cascading similar triangles. It is no problem to generalize this example to an arbitrary n number of subdivisions. Lastly, if we consider then end points of line L' to be of unit length, then the length of each segment is $1/5^{\text{th}}$.

It is intuitively clear that we can continue to subdivide an already subdivided segment, and to continue this process indefinitely. This means that we can construct points on a line which are as close together as we please to get them, to within a certain approximation. But the most important conclusion we must draw from this indefinite subdivision is this: That, at a minimum, **the number system we wish to construct on the Euclidean line consistent with geometric constructions must be a dense set, and, if we have already chosen the 0 and 1 points, then we can by this technique, in principle, construct any positive rational number.**

A set is said to be dense if between any two distinct points of the set there is at least one more point of the set. The set of integers is not a dense set, of course. By using these "straight-edge and compass" operations we can construct any "rational number." First, we decide on a unit length. Then, to create a line segment of length *m/n*, we form a new line segment of length *m* by taking *m* copies of the unit laid end-to-end along a straight line, which is then subdivided into *n* parts by the method just presented.

Note that in Fig. I.4b, the angle between the line *L'* and the line segment *L* is irrelevant as long as it is different from 0° or 180°. We have chosen the angle to be about 30°, only because it is esthetic and creates a compact diagram. A figure so constructed will be referred to as a "wedge."

In fact, we eventually want to have a number system that can deal with any magnitude that a geometer can construct, even if the method of construction goes beyond "straight-edge and compass" operations. One such magnitude is the circumference of a circle, say for simplicity, a circle of unit diameter. Now, there is no problem constructing a circle of unit diameter by "straight-edge and compass" operations, but how do we measure the circumference of such a circle? Mark a single reference point on the circle and roll the circle along a line without slipping. Every time the reference point on the circle touches the line, mark that point on the line. The magnitude of the circumference of this circle is defined to be the distance between two adjacent markings on the line. This is the number π, which we know to be irrational, and we've just created intervals of length π. We can construct another irrational number by constructing a square whose sides are unit length. Pythagorus's theorem tells us that the length of its diagonal is $\sqrt{2}$, which is shown to be irrational by simple arguments. Obviously, our number system should contain these numbers as well.

II. Reality according to the "wedge"

The present section is a more formal and detailed exposition of the material in the Introduction. The wedge will be the central artifice of exploring the number system on the Euclidean line.

There are three "gauges" (or rather, free parameters) that must be chosen to build a number system on top of a "naked" Euclidean line: First, a special point on the line must be chosen as the origin, which we will label as '0'. It doesn't matter which point is chosen for this honor, but one must be decided on. (We will refer to this choice as the "origin gauge.") The second gauge to fix is the 'unit' length, which is determined by choosing any other point on the line and calling it '1'. Now, this choice has fixed another gauge in the process: The choice of the origin point has divided the Euclidean line into two parts (i.e., two rays); by choosing the unit point we have also determined the "positive" side of the line to be that ray which contains the unit point, leaving the other ray to be the "negative" side. Figure II.1 shows the state of the process so far.

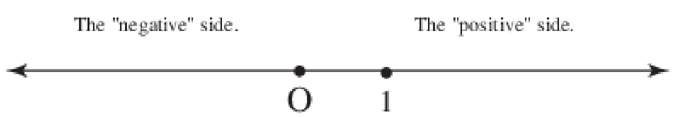

Figure II .1. The first three gauges have been fixed.

Perhaps you have noticed that there is another gauge to fix that we haven't mentioned so far, which is the orientation of the Euclidean line in the plane; or in other words, the "angle" that the Euclidean line makes relative to a horizontal line. Actually, the number system we will build on top of the Euclidean line is dependent only on the intrinsic geometry of the line, and not on its extrinsic geometry—that is, not on its orientation in the plane, or in space, etc. So, for convenience, we will always choose this extrinsic angle gauge to be zero, which fixes the Euclidean line in the horizontal position.

A word should be said here about the concept of angles and their measure in the context of the present investigation. First, we define the notion of a "linear element." *A linear element* is either a line segment, a line, or a ray. The *extension* of a linear element is defined as the replacement of a linear element by a line that contains the linear element. So, a line gets extended to itself and a line segment and a ray get extended to a line having the same orientation in the plane as the original line segment or ray. Now, an *angle* is a relation between any two linear elements in the plane, such that if the linear elements or their extensions intersect in a single point, the measure of the angle between the two linear elements is said to be nonzero; otherwise, the measure of their angle is zero. Lastly, the angle relation is defined between any linear element and itself, its angular measure being defined as zero. We shall assume all facts relating to the measurement of angles in Euclidean geometry as applying to our constructions since we have embedded our constructions in the Euclidean plane; however, we will not need to single-out any other special angles except for the zero angle, the straight angle (180°), and the "nonzero" angle. Most notably, we will not have reason the make much use of the "right" angle, except, perhaps, when using the Pythagorean Theorem or some trigonometric function.

We are now ready to construct our first number system—the *Natural numbers*. In deference to modern convention in mathematics, we will define this set of numbers abstractly as the ordered set N = {0,1,2,3,...}, which is ordered by magnitude. Note that this set starts with the number zero, not with the number 1. Thus, we can dispense with the term "whole numbers" as redundant. We will also use a derivative set of the Natural numbers, called the *counting numbers*, defined as N\{0} = {1,2,3,...}. (Note that the notation A\B, where A and B are sets, denotes a sort of set subtraction, meaning that A\B contains all the elements of A except for the elements of A∩B.)

Our geometric construction of the natural numbers will entail the association of specific points on the Euclidean line with the elements of the set {0,1,2,3,...}. Now, we simply set pointers of a compass, one to the zero point and the other to the 1 point, on the Euclidean line, and then begin to mark off equal intervals along the Euclidean line moving in the positive direction only. Figure II.2 shows what we get.

Figure II.2. The Natural numbers are constructed (in part).

Obviously, we can extend the natural numbers to the *Integers*, Z = {...,-3,-2,-1,0,1,2,3,...}, by "marking-off" points to the left as well (see Fig. II.3.).

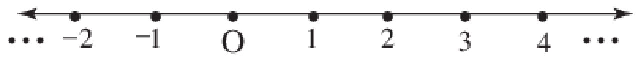

Figure II.3. The Integers are constructed (in part).

Now, we draw a new Euclidean line through the origin point at some nonzero angle, giving us the “wedge” we see in Fig. II.4a.

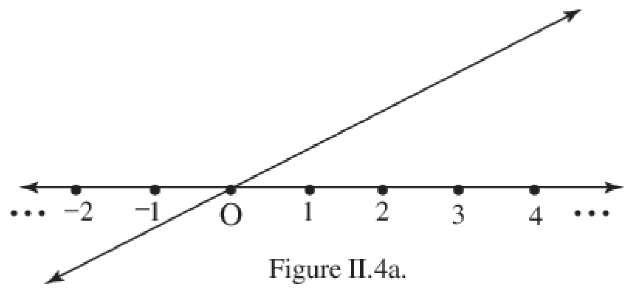

Figure II.4a.

What we want now is to reproduce the integers on this new line the same as the old line. We have at least two ways to proceed: The first is to simply reproduce the steps we took to put the integral points on the first line. The second way is to “circularly project” the points from the old line to the new. We can do this with just a compass operation, as seen in Fig II.4b.

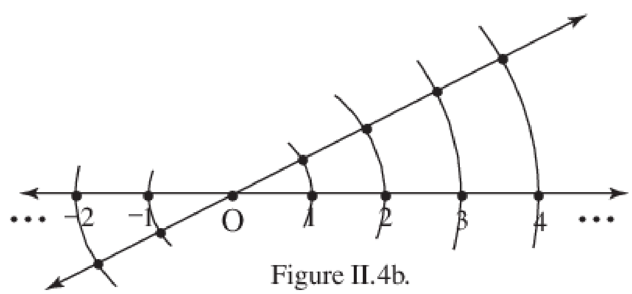

Figure II.4b.

Let’s introduce some convenient terminology at this point. We shall henceforth refer to the original Euclidean line with its hitch-hiking number system on top as the “horizontal” or “level” line, and the new Euclidean line with its number system attached as the “inclined” line. Now, as of Fig. II.4b we have ‘circularly projected’ or ‘copied’- the number system from the horizontal line onto it. **The numbers are the points themselves**. The numerals {... ,‘-3’, ‘-2’, ‘-1’, ‘0’, ‘1’, ‘2’, ‘3’, ...} merely label the numbers. As long as the circular arcs are displayed to indicate which points are circularly projected onto the inclined line, we don’t really need to place any numerals on the inclined line. Just the same, sometimes we will place numerals on the inclined line. Quite often the labeling of points on the inclined line or the manifestation of a circular projection onto it will be selective to reduce needless cluttering of the figure.

For the present time, let's concern ourselves only with the positive region of the double wedge figure, displayed below:

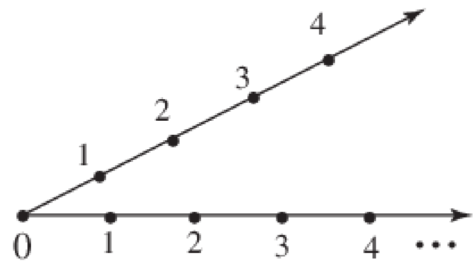

Figure II. 5. The positive sector of the wedge.

Now we need a systematic method to identify and label the points in between the natural numbers. But before we can extend the natural numbers to the positive rational numbers was must first digress a moment to learn how to do multiplication and division in the wedge. In Fig. II. 6, we find a method of multiplication based on similar triangles.

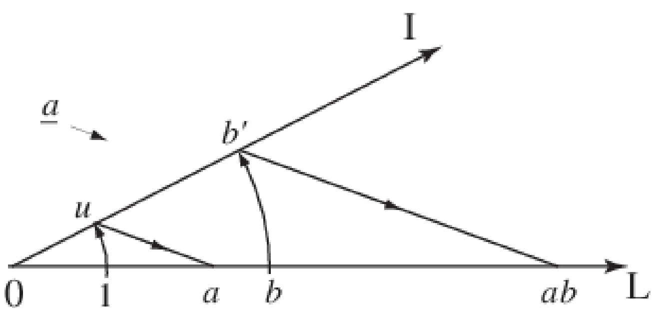

Figure II. 6. Multiplication of the number b by the number a by using similar triangles. The number 1 is circularly projected onto the inclined line, as is also the number b. Δ u 0 1 is similar to Δ b' 0 ab. The line segments inside the wedge are parallel.

Now, the caption of Fig. II. 6 already explains how to do the multiplication using similar triangles. And before going on to the "map-wise" view of the figure, let's be sure to note the important feature of this multiplication: which is that the point *ab* was not measured along the **L**-axis, but rather was constructed without knowing the value of the product *a* times *b*. In other words, the multiplication we have done here is analog, not digital: The line segment $\overline{Oab}$ has been constructed by geometry, rather than by measuring *a* and *b* against the unit length and then multiplying those two numbers together on a calculator, say.

To get the mapping point of view, consider the set of all triangles in the wedge similar to $\Delta u0a$. Multiplication by *a* not only takes *b'* to *ab*, it also takes every other point *x'* on **I** to the point *ax* on **L**. Thus, we have a natural association of points on **I\{0}** to points on **L**. If we take the point set **I** as the domain of a map and the point set **L** the range of the map, we can define a one-to-one mapping of **I** to **L,** if we remove the zero point from the domain of the map. (By our construction, the zero point on **I** would be mapped to the entire line **L**.) The main advantage of establishing a one-to-one map between **I** and **L** is to allow for inverse maps, which will correspond to division.

Of course, we will want our number system to include ordinary addition of numbers. So that for all *x*, *y* ε **L**, *x* + *y* ε **L.** Another way to state this is that (**L,+**) is closed under addition. We will also want to be commutative and associative, as usual. This makes (**L,+**) a group, if we include additive inverses. But this deficiency will be eliminated when we extend **I** and **L** to have negative numbers. As usual we want multiplication to be commutative, associative, and distributive.

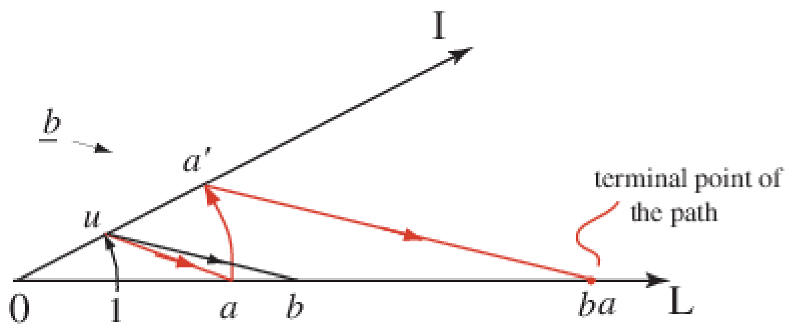

Figure II. 7a. Multiplication of the number *a* by the number *b* by using similar triangles. The number 1 is circularly projected onto the inclined line, as is also the number *a*. $\Delta u01$ is similar to $\Delta a'0ba$. The line segments inside the wedge are parallel. The red connected contours represent a "pathway throught the wedge."

Fig. II. 7a shows that from the viewpoint of a **geometrical construction**, *ab* is not the same as *ba*, but *ab* = *ba* as locating the same terminal point in the two constructions. The "proof" of this is merely to construct *ab* and *ba* on the same wedge diagram. Fig. II. 7a shows a "pathway through the wedge" in red. Fig. II. 7b shows an alternative "pathway through the wedge" in red.

Let's introduce some new terminology. From the theory of ratios we have the following definitions: Let *a,b,c* be three numbers in an appropriate commutative number system. A *ratio* or a *quotient* of "*a* to *b*" is said to occur whenever there exists a number *c* such that $cb = bc = a$. The value of the ratio is simply $c = b^{-1}a = ab^{-1}$. The numeral forms to represent the "ratio of *a* to *b*" are "*a*/*b*" or "*a*÷*b*" or

$$\frac{a}{b}.$$

These three numerals are called *fractions*. A *proportion* is a statement that equates two ratios, such as the statement

$$\frac{a}{b} = \frac{c}{d}.$$

The “proof” of the associative property of multiplication will be left as an exercise for the interested reader, but we will help by providing a figure for the product $a(bc)$.

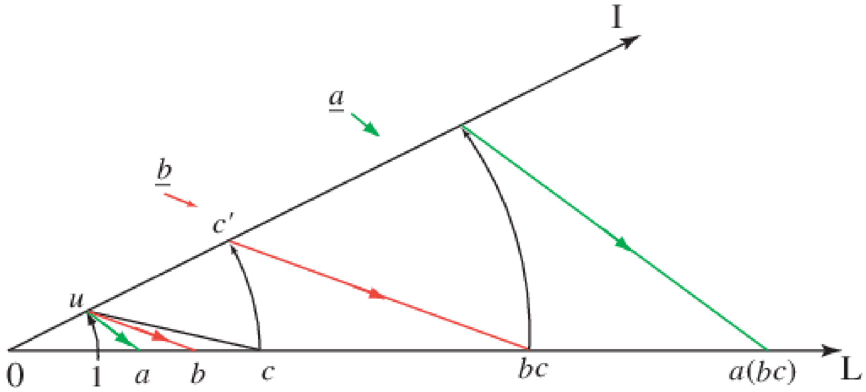

Figure II. 9. Associative multiplication of the number *bc* by the number *a* to get *a*(*bc*).

Thus, we have demonstrated that we include associative multiplication to our number properties, making our number system on **L** a field.
Now let’s further investigate the nature of the mapping $\underline{a}$. Let x, y ε **L**, and let $\underline{a}$ ε Map(**I**,**L**), where Map(**I**,**L**) means a linear map from **I** to **L**. Another way to state this is that $\underline{a}$: **I** → **L**, given by $\underline{a}\,(x) = ax$, $\underline{a}\,(x+y) = ax + ay$. Now let λ ε **scalars** be a number used for digital multiplication, that is, for scaling between similar triangles. We assume that $\lambda a = a\lambda$ for all a ε **L** and for all a ε **I**, making the digital multiplication commutative. Let’s further assume that digital multiplication in the scalars is associative. With all these properties we have established, we are now ready to show that $\underline{a}$ is a linear map.

$$\begin{aligned} \underline{a}\,(\lambda x+y) &= a(\lambda x+y) = a\lambda x+ay = \lambda ax+ay \\ &= \lambda\,\underline{a}\,(x) + \underline{a}\,(y) \end{aligned} \tag{1}$$

What happens if we take the set of all lines parallel to the line through the unity of the inclined line and some arbitrary point on the level line, say point *a*. If we circularly project *a* up to the inclined line, it will meet one of the set of parallel lines. And this parallel lines connects that projected point back onto the level line to the point $\underline{a}\,(a) = a^2$. By continuing this process we can get to the $(n–1)$-st power of *a* on the level line and one more circular projection and then application of the $\underline{a}$ on that point brings us to the *n*-th power of *a* on the level line. Fig. II.10 show the first few powers which can be located by this method.

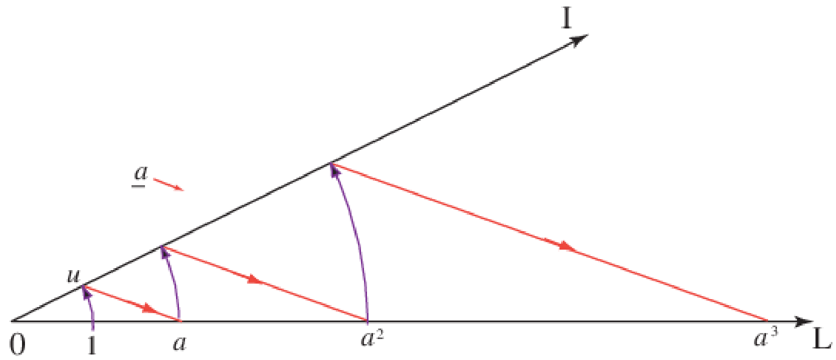

Figure II. 10. Recursive analog multiplication. The RAM diagram.

From the viewpoint of the map interpretation, the circular projection does more than just project the label of points from the level line to the inclined line. It also takes us from the range space back to the domain space on which the map is defined.

III. Practical uses of the "wedge"

First, given $b < a$, use the Wedge to show that $a^{-1} < b^{-1}$. The geometrical construction is given in the figure below.

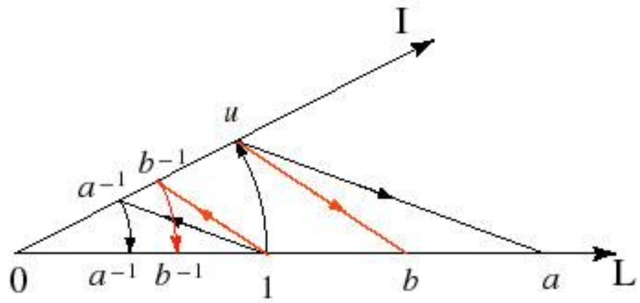

Fig. IIIa. The construction and comparison of inverses in the Wedge.

One could even find the inverses of real numbers by setting the number one wants to find the inverse of to *a* on the wedge above, and then locating a^{-1} and measuring from 0 to a^{-1}. Just don't expect much accuracy.

Next, find by construction the real number $5^{2/3}$. To accomplish this, we'll use the RAM diagram--RAM for Recursive Analog Multiplication.

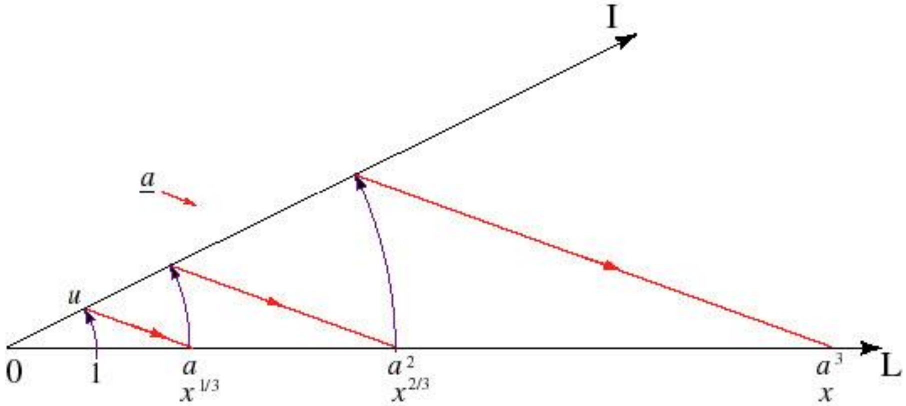

Fig. IIIb. The construction of the cube root and two-third roots of *x*.

Just set *x* = 5 to get the cube root and two-third roots of 5.

IV. Conclusion

Thus, we have shown how to construct a model of the real numbers on a number line, starting from simple beginnings and building up. Further uses of this "wedge technology" will be found in the follow-up paper on high school geometry problems.