Euclidean Geometry Problem Solving Through Symmetry Analysis: Transformations and Embeddings

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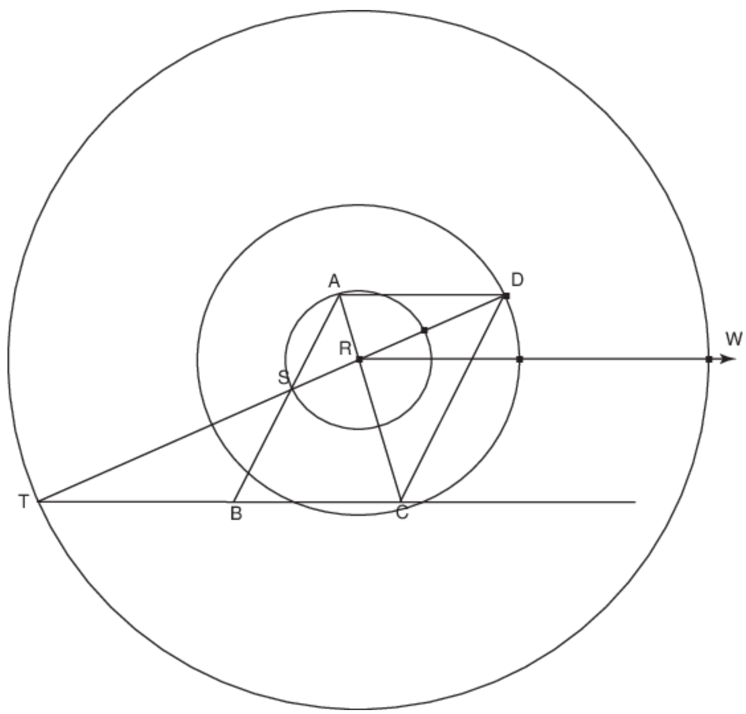

This revised paper demonstrates what can be accomplished toward making high school geometry easier by using the method of symmetry analysis. This analysis uses:

- constructions to create highly symmetric figures in which the original geometric figure from a particular problem can be embedded and then easily solved,
- transformations of the figure into equivalent but more highly symmetrized figures
- advanced analytic tools built out of geometrical constructions, and
- a systematic set of specific heuristics.

The diagram in Fig. 1 was found in a standard high school geometry textbook: *ADCB* is a parallelogram. The problem also included an algebraic relation given as something to verify. We, however, will use our modern computer skills to help us formulate the conjecture and then to go on to verifying the conjecture algebraically. The only hint we will give at this point is that the conjecture involves the lengths of line segments lying on segment $\overline{TD}$.

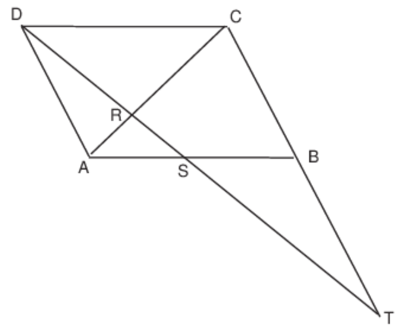

Figure 1. *ADCB* is a parallelogram.

Our first thought to solving this problem is to use the wedge technology that we have cultivated so well. To that end we redraw the figure as in Fig. 2.

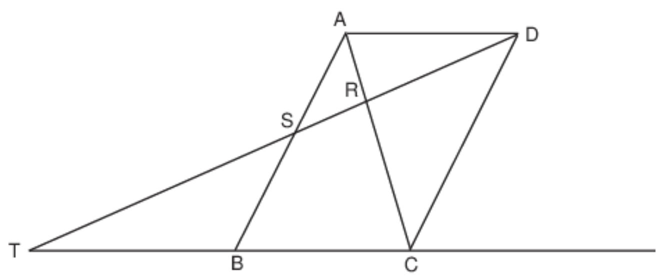

Figure 2. We redraw the figure to leverage what we already know: the Wedge technology.

Now, it's obvious that ∠DTC forms a wedge, and we will be using this wedge later. But for now we will add the ray *W* with vertex at point *R* and point it off to the right to our current figure. The reason we are doing this is because of the hint we were given about the conjecture concerning lengths of line segment lying on segment $\overline{TD}$. We will use the point *R* as a centrum (i.e., a center point about which two or more concentric circles will be drawn) to project some of the points off the $\overline{TD}$ segment. This we do to the points *T*, *S*, and *D*. Just which point or points we will project off the line we will determine after the new construction. So, let's go ahead a draw the projection circles and display the result in Fig. 3.

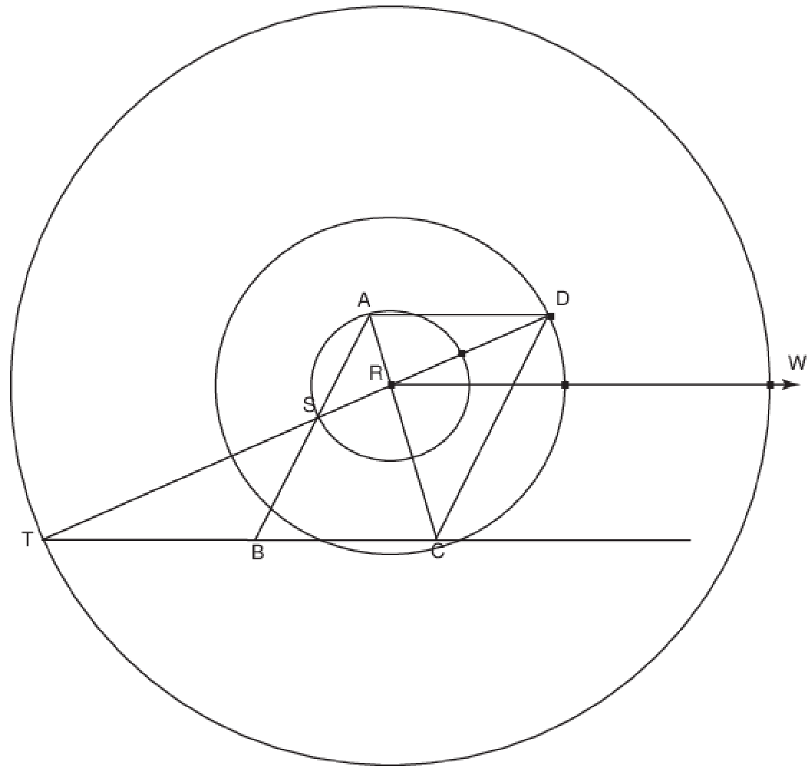

Figure 3. The points *T*, *S*, *D* manifesting their "circles of influence" by being circularly projected about centrum *R* into the wedge

From Fig. 3 we cull out just the part of interest at this moment, which is displayed in Figure 4.

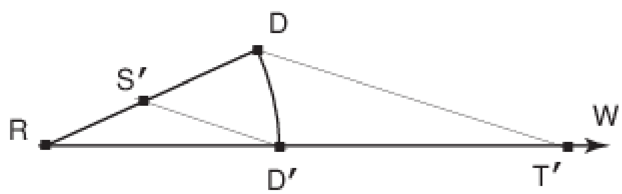

Figure 4. A subfigure from Fig. 3 showing the "magic" secondary wedge we have constructed so easily.

The most obvious conjecture to make about Fig. 4 is that $\overline{S'D'}$ is parallel to $\overline{DT'}$ (denoted as $\overline{S'D'} \parallel \overline{DT'}$). If that's true, then we can conjecture the following algebraic relation

$$\frac{RS'}{RD'} = \frac{RD}{RT},$$

Where we used that $RT' = RT$. And, since $RS' = RS$, $RD' = RD$, then

$$(RD)^2 = RS \bullet RT. \tag{1}$$

Thus, we have arrived at the starting equation given to the student in the textbook. So how should we proceed?

Unfortunately, we don't have an obvious means to prove that $\overline{S'D'} \parallel \overline{DT'}$, so we can proceed to solve the problem in the usual way. The first thing we must do is to return to Fig. 2 and do some more creative interpretation and drawing.

First, we interpret ∠DTC as a wedge, meaning that we are on the lookout for any sets of parallel lines that we can manufacture in the wedge. We can sum up the Wedge's power with this quatrain:

There's secret power in the Wedge I draw,
by what I mustn't tell!
But all the other lines I draw
are always parallel!

With the interpretation of ∠DTC as a wedge we can immediately appreciate that $\overline{AB}$ is parallel to $\overline{DC}$, from the given information. But what about $\overline{AC}$? Should we mate it with a parallel line segment through the wedge? We should indeed do so AUTOMATICALLY if the line segment or its mate are relevant to the problem at hand. In our case, $\overline{AC}$ divides $\overline{TD}$ at *R* and this is probably very relevant to establishing Eqn. (1). Now, we draw a line segment through D parallel to $\overline{AC}$, hitting line $\overline{AC}$ at *E*. Thus, we arrive at Fig. 5, where the tick marks show the line segment congruences.

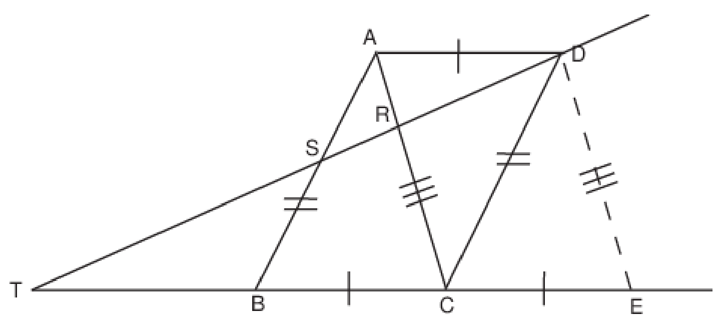

Figure 5. The point *E* is derived by construction so that $\overline{DE} \parallel \overline{AC}$. The congruencies indicated between $\overline{AD}$ and $\overline{TE}$ should be interpreted as referring to complete segments lying between $\overline{AD}$ and $\overline{TE}$, such as *AB* = *DC*.

We are finally ready now to begin writing down algebraic equations, but we must do so conservatively to avoid being swamped by redundant or useless information. Remember that every set of parallel lines through a wedge generates a **chained proportion.** For example, consider the generic wedge in Fig. 6.

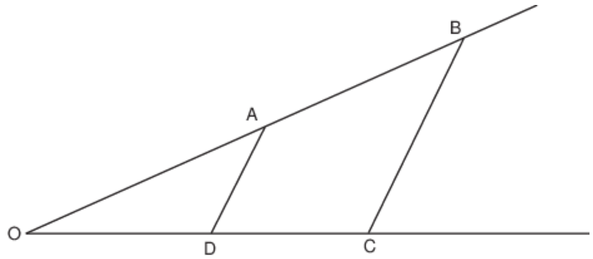

Figure 6. A generic wedge to demonstrate **chained proportions**.
Here $\overline{AD} \parallel \overline{BC}$. In a generic wedge one segment has one length that is strictly less than and another that is strictly greater than the lengths of all other segments on the inclined and level lines. In our case this is the segment *OD* for the shortest and *OB* for the longest. Of the four points *A, B, C, D*, then, *D* and *B* are called the "extreme" points, leaving points *A* and *C* as the "mean" points of the four.

In Fig. 6 $\overline{AD} \parallel \overline{BC}$, from which we know that $\Delta OAD \sim \Delta OBC$. And from this we get the following chained proportions:

$$\frac{OA}{OD} = \frac{OB}{OC} = \frac{AB}{DC} = \frac{AD}{BC}, \tag{2}$$

where the first three ratios are from line segments on the inclined and level lines. The last ratio comes from considering the two segments $\overline{AD}$ and $\overline{BC}$ "through the wedge." Note also that the number of chained equations gets larger for each additional parallel line placed through the wedge.

The next part of the analysis concerns what to call all other relations on line segments besides those of proportions. One type occurs when a line segment is subdivided into parts, from which we form an equation by equating the total to the sum of its parts, such as in Fig. 6:

$$OB = OA + AB. \tag{3}$$

This type of equation, together with inequalities that can be formed on line segments, we refer to as **constitutive relations**.

Alright, we are now ready to **formulate a plan** of culling information from Fig. 5.

1) We have two pairs of parallel line segments, generating two independent sets of chained proportions. We will write these down, omitting the probably unnecessary ratios from the segments through the wedge.

2) We will choose from each chained proportion a "simple" equation relating the most relevant line segments from each chain.

3) From this pair of simple equations we will eliminate the most irrelevant line segment. If two or more seem just as irrelevant, we can randomly choose any one of them.

4) We will bring only those constitutive relations that are needed.

SOLUTION:

STEP 1):

$$\frac{TS}{TB} = \frac{TD}{TC} = \frac{SD}{BC} \;, \tag{4}$$

$$\frac{TR}{TC} = \frac{TD}{TE} = \frac{RD}{CE} \;. \tag{5}$$

STEP 2):

$$\frac{TD}{TC} = \frac{SD}{BC} \;, \tag{6}$$

$$\frac{TR}{TC} = \frac{RD}{CE} \;. \tag{7}$$

STEP 3): Of all the line segments represented in Eqn's (6) and (7), $\overline{TC}$ looks as irrelevant to establishing Eqn (1) as imaginable. So, we will solve (6) and (7) each for *TC* and then use the transitive property of equality.

$$TC = \frac{TD \cdot BC}{SD} \;, \tag{8}$$

$$TC = \frac{TR \cdot CE}{RD} \;. \tag{9}$$

Therefore,

$$\frac{TD \cdot BC}{SD} = \frac{TR \cdot CE}{RD} \;. \tag{10}$$

But $BC = CE$, yielding

$$\frac{TD}{SD} = \frac{TR}{RD} \;. \tag{11}$$

There are a couple observations to make at this point. The first is that the wisdom to mate $\overline{AC}$ with $\overline{DE}$ is quite clear now. The second is that Eqn. (11) seems just as important as Eqn. (1).

STEP 4): Now we bring in any needed constitutive relations. Keep in mind an important lemma

$$\frac{a+c}{b+d} = \frac{c}{d} \qquad \text{if and only if} \qquad \frac{a}{b} = \frac{c}{d}\ . \tag{12}$$

Since

$$TD = TR + RD, \tag{13}$$

and

$$SD = SR + RD. \tag{14}$$

Substituting these into the left-hand side of (11) gives

$$\frac{TR+RD}{RD+SR} = \frac{TR}{RD}. \tag{15}$$

Then matching (15) to (12) gives us

$$\frac{TR}{RD} = \frac{RD}{SR}. \tag{16}$$

Finally, since $SR = RS$, and then by cross multiplying, we get Eqn. (1).

It's fair to say that the ability to conjecture Eqn. (1) without the aid of the wedge construction to produce Fig. 4 is highly unlikely for most of us. The question is why is the splitting of the line $\overline{TD}$ so useful a trick. I think it's the nature of the original problem, which was to find a relationship among line segments that all lie on the same line. There's not much geometry going on within a single line. Or perhaps there's a lot of 'geometry' happening on $\overline{TD}$ but it's hidden from sight. The wedge construction used in Fig. 4 just helps to bring this hidden geometry to the light of day where we can easily see it.

Our wedge construction to arrive at Fig. 4 is metaphorically like "splitting" the line $\overline{TD}$ to produce a wedge, taking some points on the new line and leaving others on the old. This trick of splitting an overburdened line to relieve it of so many line segments or points is so important that it deserves its own quatrain to commemorate it:

When we split the Atom
  some thought that really fine.
But no one ever dared believe
  we could split the Real Line!

Certainly, the wedge construction is a tool to help us mentally cope with hidden or complicated structure.

Much importance has been made of efficient heuristics in this paper because so few textbooks deal with it in a systematic way, if at all. The biggest problem with finding information from Fig. 5, say, as an example, is that if you write down all possible equations that could be written from it, you'd have maybe 50% of the resulting set of relations as redundant information. With so much redundant information and lacking a logical plan to deal with it, the chances of "going in circles" while groping for a solution is quite high.

Now for our second problem.

In the figure below, $\overline{AN}$ is perpendicular to $\overline{BC}$ and ∠BAC is a right angle. Show that

$$(BA)^2 = BC \bullet BN \,. \tag{17}$$

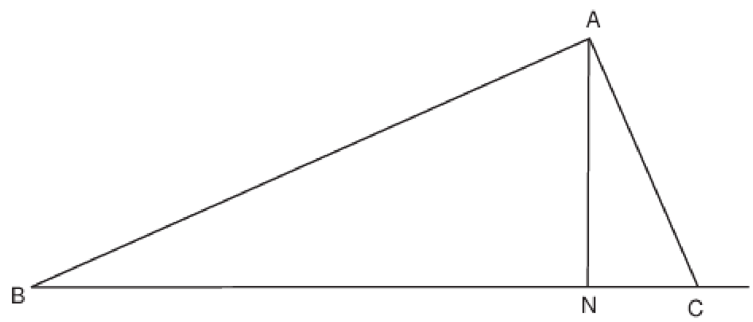

Figure 7. In ΔBAC, $\overline{AN}$ is perpendicular to $\overline{BC}$ and ∠BAC is a right angle.

To proceed, of course we embed the figure in a wedge and then add two points by circularly projecting points A and N to get:

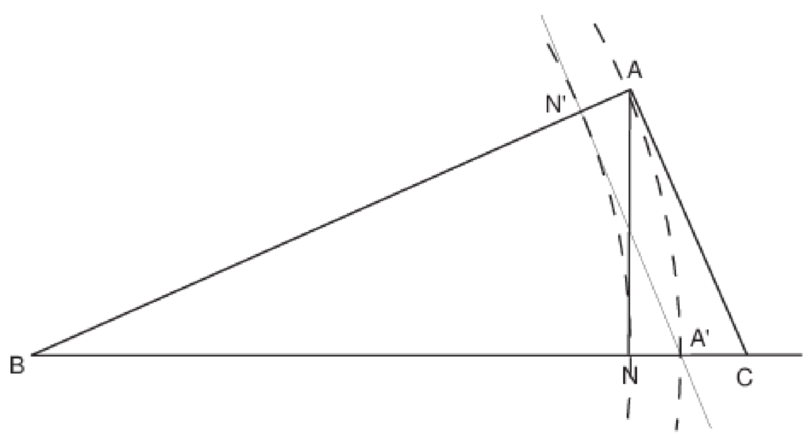

Figure 8. *A* and *N* are projected and $\overline{N'A'}$ is drawn.

On just the visual image of lines $\overline{N'A'}$ and $\overline{AC}$ we could conjecture that they are parallel, and thus that Equation (17) follows immediately. Now we could prove that they are parallel by proving that $\Delta BAC \sim \Delta BN'A'$. But we have a more accessible result to prove: $\Delta ABN \cong \Delta A'BN'$. Well, since *B* is a centrum we have that $BN = BN'$, $BA = BA'$, and it's obvious that $\angle ABN = \angle A'BN'$, thus by the SAS postulate: $\Delta ABN \cong \Delta A'BN'$. And now that we know these two triangles are congruent, we can equate corresponding angles in them, namely, $\angle BN'A' = \angle BNA = 90°$. Therefore, $\angle BN'A' = \angle BAC$, establishing that $\overline{N'A'} \parallel \overline{AC}$. And we are finished.

Although we did not need a wedge to prove this result, it did facilitate the proof not to mention facilitating the conjecturing of the theorem itself. The way to remove the perceived mysteriousness of geometry from the minds of highschool students is to facilitate them in conjecturing the very theorems they must then prove.

Another Wedge tool.

Thus far, we have stressed the usefulness of using parallel lines through the wedge, but now we will investigate another useful construction within it. Consider the following figure:

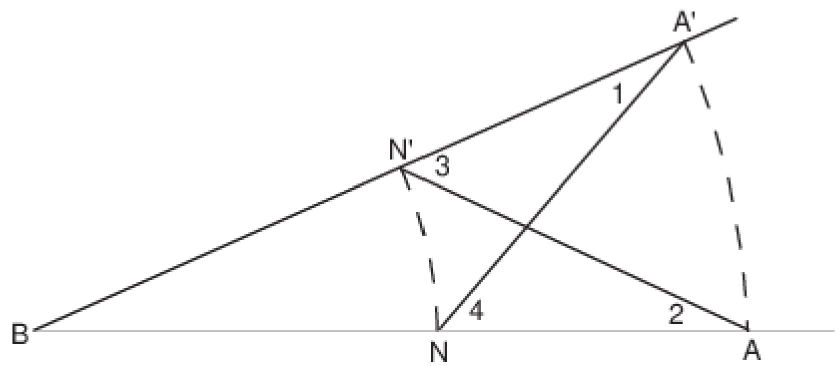

Figure 9. By circularly projecting points *A* and *N* from the level line to the slanted line we have created a number of angular relations to be sorted out.

By the SAS postulate we can easily see that $\Delta NBA' \cong \Delta N'BA$. So we can also conclude that $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$. We shall refer to this as Wedge Theorem 3.

Now for preparation for future problems: The calculus of Supplementary & Complementary forms

Now for a new and convenient calculus of complementary and supplementary forms. Assume that *a* is any real number, then we define $\bar{a}$ by

$$\bar{a} \equiv \pi - a \qquad (18a)$$

which simply defines a symbol to represent the generalized supplement of angle a in radian measure. From (18a) we get the following results:

$$a + \bar{a} = \pi \qquad (18b)$$

$$\bar{\bar{a}} = a \tag{18c}$$
$$b = \bar{a} \quad \text{if and only if} \quad \bar{b} = a \tag{18d}$$
$$(\bar{a} + \bar{b}) - \overline{a+b} = \pi \tag{18e}$$
$$(\bar{a} + \bar{b}) = 2\pi - (a + b) \tag{18f}$$
$$\overline{\bar{a}+\bar{b}} = -(\overline{a+b}) \tag{18g}$$
$$\overline{\bar{a} \pm b} = a \mp b \tag{18h}$$
$$\bar{a} = a \quad \text{if and only if} \quad a = \pi/2 \tag{18i}$$

The proof of the equations in (18) are easy and left for the reader. Obviously, the relations are useful for straight angles but also for triangles. For example, let a, b, c be the interior angles of any triangle, then

$$a + b + c = \pi \qquad \text{implies that} \qquad \bar{a} = b + c \tag{19}$$

So, what are the sums of the exterior angles of the same triangle? Well, each exterior angle is the supplement of its adjacent interior angle, so

$$\bar{a} + \bar{b} + \bar{c} = 3\pi - (a + b + c) = 2\pi \tag{20}$$

Although the relation $\bar{a} = b + c$ is equivalent to the Exterior Angle formula for a triangle, it was derived from a more general equation.

Of course, we have a similar notation for complementary angles.

$$\tilde{a} \equiv \pi/2 - a \tag{21a}$$

which simply defines a symbol to represent the generalized complement of angle a. From (21a) we get the following results:

$$a + \tilde{a} = \pi/2 \tag{21b}$$
$$\tilde{\tilde{a}} = a \tag{21c}$$
$$b = \tilde{a} \quad \text{if and only if} \quad \tilde{b} = a \tag{21d}$$
$$(\tilde{a} + \tilde{b}) = \pi/2 + (\mathrm{a} + \mathrm{b})^{\sim} \tag{21e}$$
$$\tilde{a} + \tilde{b} = \overline{a+b} \tag{21f}$$
$$\bar{a} + a = 2(b + c) \quad \text{if and only if} \quad c = \tilde{b} \tag{21g}$$
$$\bar{a} + a = 2(b + \tilde{c}) \quad \text{if and only if} \quad c = b \tag{21h}$$

Now (21g,h) may look like just a convoluted ways to state the equivalent of (21b), but they do occur frequently enough to justify their presence in the list.

A word about notation is needed here: I will from here on distinguish between an angle and its measure. So, if $\angle 1$ represents angle 1, then $m\angle 1$ represents its measure, a real number.

Now for our third problem.

This next problem is easy but also illustrative. Consider the figure below.

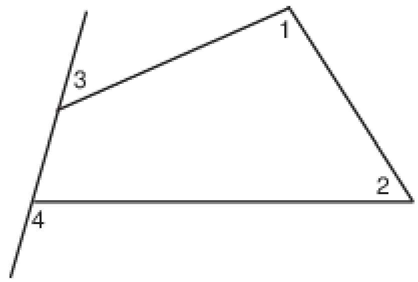

Figure 10. An arbitrary quadrilateral.

With respect to the angles depicted in Fig. 10, show that m∠1 + m∠2 = m∠3 + m∠4. Using our calculus of supplementary forms we get

$$m\angle 1 + m\angle 2 + m\angle \bar{3} + m\angle \bar{4} = 2\pi \qquad (22)$$

Now , since $m\angle \bar{3} = \pi - m\angle 3$, and $m\angle \bar{4} = \pi - m\angle 4$, then we get

$$m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4 \qquad (22)$$

Another tool.

The wedge is strong in dealing with parallel lines, but there are plenty of situations that present nonparallel lines for consideration. Of course, nonparallel lines intersect in the plane and thus form a wedge, but this may not be useful in all problems. When nonparallel lines cut through a wedge they don’t form similar triangles but they do form relatable triangles. Consider the form below which I dub a “butterfly.”

In Fig. 11 we see the full wedge cut by two nonparallel lines forming a sort of “butterfly” configuration. Now the two triangles formed are not usually similar, but they do carry an important relation between them. At the vertex *b* which joins the two wings of the butterfly are so-called vertical angles, which are equal. So, their supplements are equal.

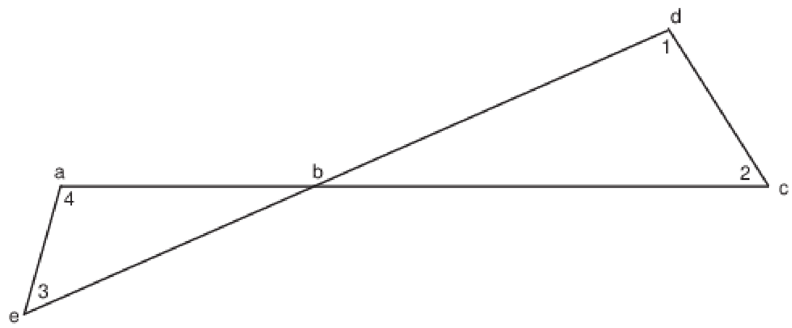

Figure 11. The “Butterfly” form.

And from this we get the **Butterfly Theorem**:

$$m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4 \qquad (23)$$

From this we have an easy corollary, called the **Similar Butterfly Corollary**: If any nonvertical angle of one wing of a butterfly is equal to any nonvertical angle of the other wing, then the remaining corresponding nonvertical angles are equal, and the two wings are similar by AAA Postulate.

Now we draw a line parallel to $\overline{ae}$ through the right wing of the butterfly to get

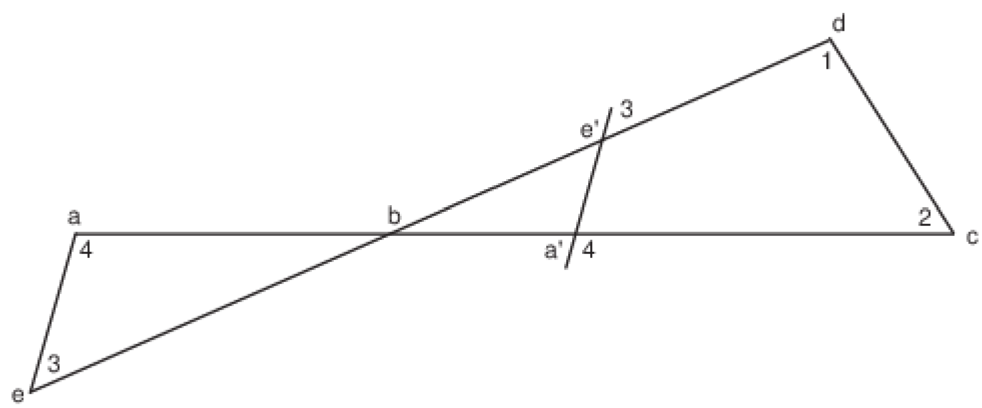

Figure 12. The “folded butterfly” *bdcbe’a’*.

We see in *a’dce’* of Fig. 12 the same quadrilateral as in Fig. 10. Were this transformation of Fig. 11 to get Fig. 12 done with the aid of appropriate geometry software (as a continuous motion of line $\overline{ae}$ to line $\overline{a'e'}$), the resulting proof could easily be considered as a “visual proof.”

Yet another tool.

Anyone who has taken highschool geometry knows that the circle is a figure capable of generating very complicated problems to solve. But I have in mind another interpretation of the circle: It is a very powerful tool in helping to solve problems. The simplest feature to demonstrate for the circle is its ability to effortlessly generate isosceles triangles. Just take any two points on the circle that aren't collinear with the center of the circle and you've got an isosceles triangle, as in Fig. 13. An important symmetry of an isosceles triangle is that its base angles are equal in measure.

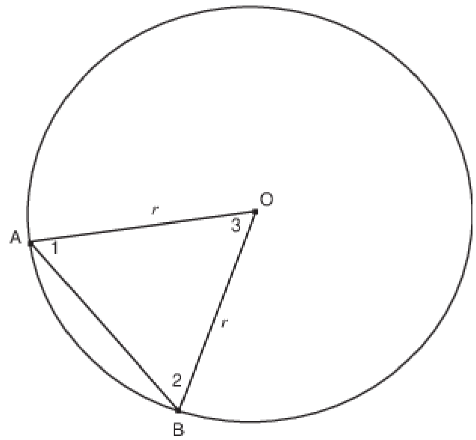

Fig. 13. Isosceles triangles made easy. m∠1 = m∠2.

Now for our fourth problem.

Using the isosceles triangles of the circle we will prove the following important theorem about inscribed angles in a circle: *The measure of an inscribed angle intercepting a minor arc in a circle is equal to half the measure of the central angle intercepting the same arc of the circle (the inscribed angle on the major arc).*

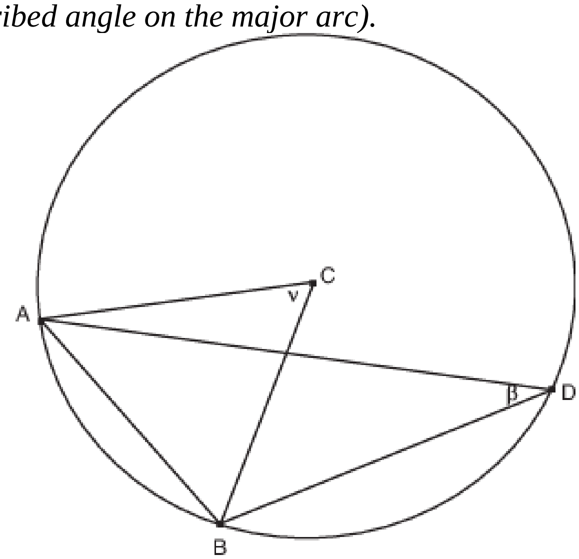

Figure 14. Central angle at *C* intercepts the same minor arc as the inscribed angle at *D*.

In Fig. 15, we have added the line segment $\overline{CD}$, making a total of three isosceles triangles from which to get relationships. Fortunately, about all we need do now is to use the Butterfly Theorem at butterfly vertex *b*, from which we get:

$$\beta + (\beta + m\angle 2) = m\angle 2 + \nu.$$

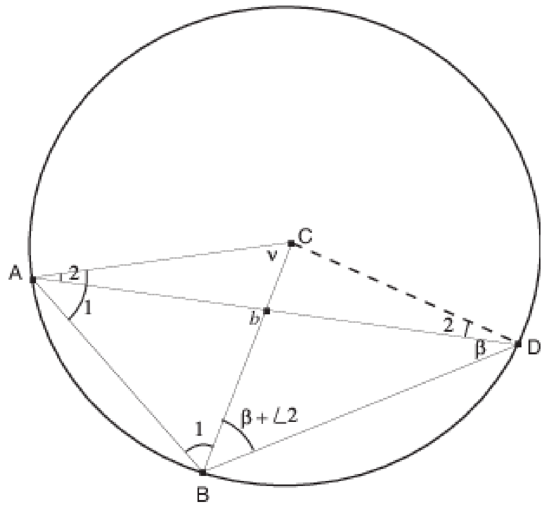

Figure 15. Combining "circular isosceles triangles" and "butterflies" can lead to compact proofs!

And this quickly simplifies to $\beta = \frac{1}{2}\nu$. Henceforth this theorem will be referred to as the *Circular (Minor) Half-Angle Theorem* (or *CHAT*). "Minor" because the inscribed angle is on the minor arc between points *A* and *B*.

Our fifth problem puts it all together!

If you thought that the synthesis of our tools on the last problem was effective, you'll love this problem. It synthesizes the wedge, circular isosceles triangles, and CHAT tools all into one proof. With that hint, maybe you can solve this next problem on your own with these tools.

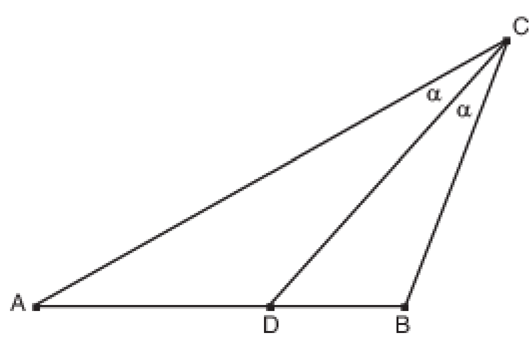

Figure 16. The "before" state.

In Fig. 16 is a triangle ΔABC in which the line segment $\overline{CD}$ has been placed such that it divides $\angle C$ into two equal angles. Show that

$$\frac{BD}{DA} = \frac{CB}{CA}. \tag{24}$$

OK, where do we start. Remember: Proportions suggest a wedge with parallel lines going through it. So, *A* could serve as the vertex of wedge *CAB*, and $\overline{CD}$ can serve as one line through it. But now we have to decide on a line parallel to $\overline{CD}$ through the wedge. This other line should contain the point *B*, but that solves only half the decision at his point. We can either draw a line through *B* which is parallel to $\overline{CD}$, or else we can draw a line through *B* which satisfies some other convenient condition and then show that this other line through *B* is parallel to $\overline{CD}$, which is what we will do this time.

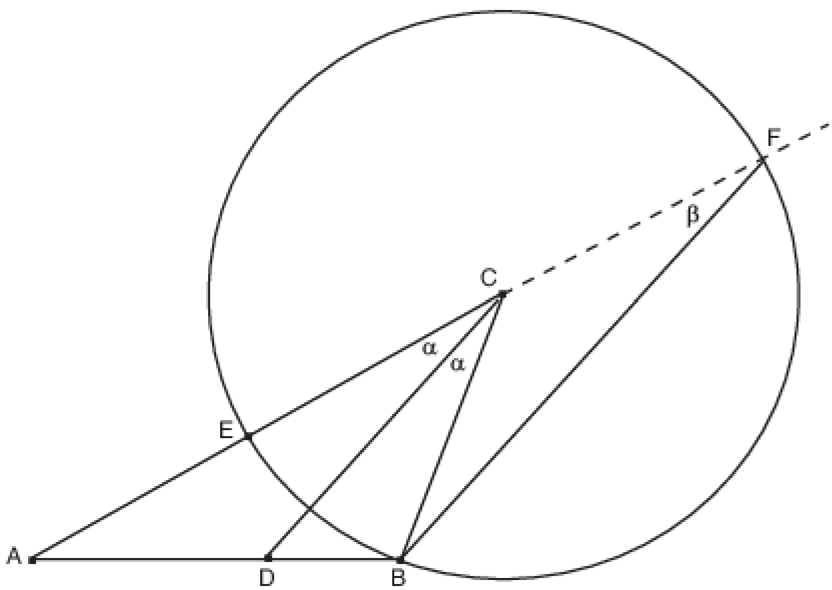

Figure 17. The "after" state—the "symmetrized" state.

Our strategy is to draw a circle of radius *CB* at point *C*, and then to extend $\overline{AC}$ until it meets the circle at point *F*. We label the point of intersection of the circle with segment $\overline{AC}$ as *E*. Now by the CHAT theorem $\beta = \alpha$ (Why?). Therefore $\overline{CD} \parallel \overline{FB}$ (Why?). Therefore

$$\frac{FC}{CA} = \frac{BD}{DA}. \tag{25}$$

But $FC = CB$, so

$$\frac{CB}{CA} = \frac{BD}{DA}, \tag{26}$$

as we needed to show.

An obvious corollary to this theorem occurs when in Fig. 16, $AC = BC$ (that is, when ΔABC is isosceles). Then ΔACD is congruent to ΔBCD (by postulate SAS), and 1) they are both right triangles and 2) $\overline{CD} \perp \overline{AB}$.

Our sixth problem also puts it all together!

In Fig. 18, we have a tangent line to a circle and an arbitrary secant line through the circle. Point C on $\overline{PA}$ is the circular projection of T about P. Formulate and prove a conjecture about the lengths of line segments on $\overline{PA}$.

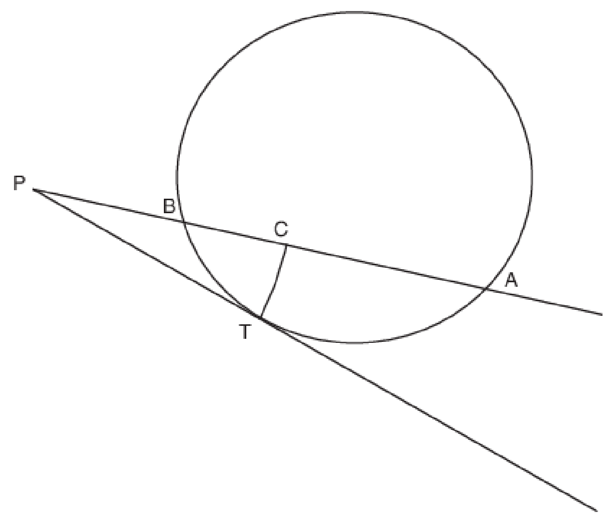

Figure 18. Initial state.

We have seen this kind of problem before. Let's guess that there exists some proportion to be cast here. Probably using P as a centrum to circularly project A onto $\overline{PT}$ and then finding relevant parallel lines through the wedge APT will give us good conjecture.

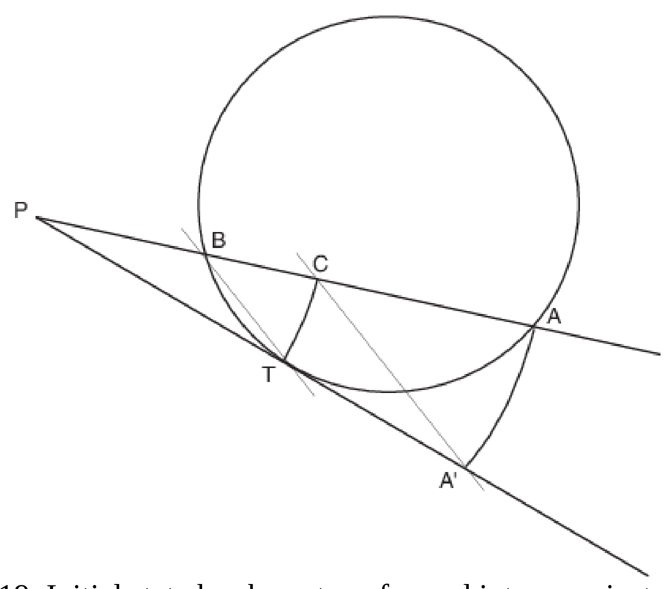

Figure 19. Initial state has been transformed into a conjecture state.

If we conjecture that $\overline{BT} \parallel \overline{CA'}$ then we can also conjecture (similar to previous problems we have done) that

$$(PT)^2 = PB \bullet PA \tag{26}$$

One way to prove $\overline{BT} \parallel \overline{CA'}$ is to show that $m\angle BTP = m\angle CA'P$. And here is the point at which one can easily go around in circles trying to prove this last equality from within the wedge alone. We have in fact culled out nearly all the information we can get from the wedge already. There's two more pieces of information to use, though. Notice that points *B*, *T*, and *A* are on the circle. Remember that a circle has an enormous amount of symmetry to use for finding relationships. If we did not already have a circle to contain these three points, we should have to construct it for its problem-solving ability.

Now, as we know, $\Delta PAT \cong \Delta PA'C$, so $m\angle PAT = m\angle PA'C$. Therefore, it is sufficient to show that $m\angle BTP = m\angle PAT$ to show that $\overline{BT} \parallel \overline{CA'}$. This is the first bit of information we needed. The second bit is that, since $\overline{PT}$ is tangent to the circle, then it is perpendicular to $\overline{OT}$, where *O* is the center of the circle. Now with this information we can drop points *P*, *A'*, and *C* from further consideration, and build isosceles triangles inside the circle. This is depicted in Fig. 20. Notice that $m\angle OTB = m\angle\tilde{1}$.

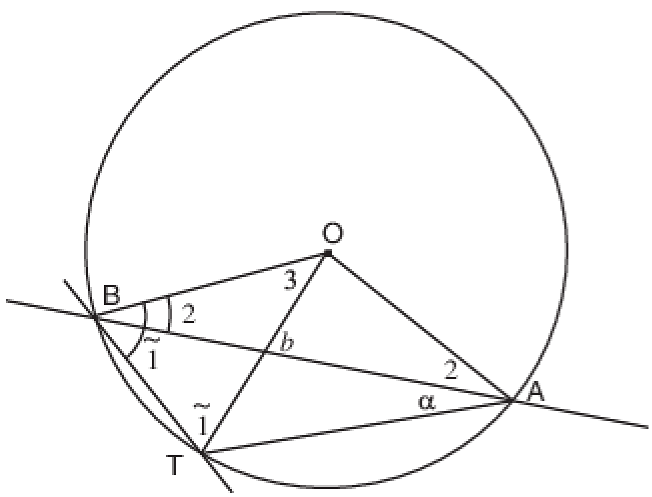

Figure 20. The figure is recast again into "solution" state.

As you can see, we have two butterflies at the point *b*, but we won't be needing either of them this time. We need CHAT applied to $\angle BAT$ to conclude that $m\angle 3 = 2\alpha$. Because ΔBOT is isosceles, we know that $m\angle\bar{3} = 2m\angle\tilde{1}$. On adding these two equations together we get

$$m\angle 3 + m\angle\bar{3} = 2(\alpha + m\angle\tilde{1}) \tag{27}$$

Applying (21h) to (27) we can conclude that $\alpha = m\angle 1$, which was to be shown.

Our seventh problem champions the Butterfly tool

From Fig. 21 we are to show that m$\angle AEB$ + m$\angle ADB$ = 2m$\angle$ AFB.

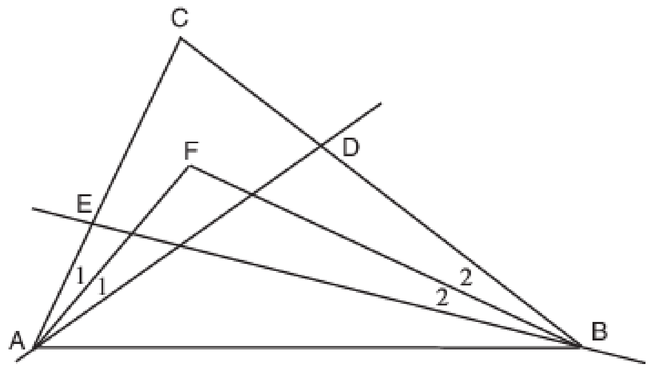

Figure 21. Is this a hard problem?

To make the solution easier we can re-label angles as in Fig. 22 below. In terms of our new labeling of angles our ShowThat equation becomes m$\angle 4$ + $\angle 5$ = 2m$\angle 3$.

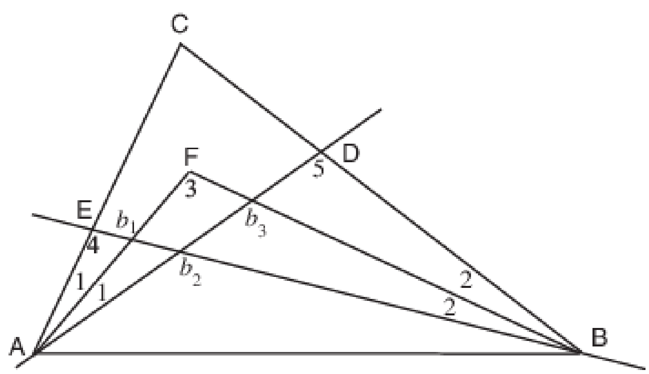

Figure 22. The angles are re-labeled.

In Fig. 22, all the b's are vertices of butterflies, and again we are in trouble of using too many butterflies to extract the information we need. We could use butterflies that may not lead to useful information or we may use too many butterflies, in which case we end up with redundant information. Let's choose just two for a start. From b_1 we get that

$$m\angle 1 + m\angle 4 = m\angle 3 + m\angle 2. \tag{28}$$

From b_2 (the big butterfly) we get that

$$2m\angle 1 + m\angle 4 = m\angle 5 + 2m\angle 2. \tag{29}$$

Now subtracting (29) from twice times (28) yields

$$m\angle 4 + m\angle 5 = 2m\angle 3, \qquad (30)$$

which is what we were to show.

We have learned three lessons so far: The first is that the tools we use and the constructions we make may transform the original problem into a simpler problem and the required algebra simpler. The second is that the constructions we have made it easier to formulate a plan of culling out the most relevant information from among the irrelevant information. Without such a plan it is easy to get overwhelmed by irrelevant and/or redundant information. And the third observation is that our constructions aid in formulating the conjectures that we then go on to prove rigorously.

The reason that some of the problems are difficult in their original state is because they are in some sense mere fragments of a larger highly symmetrical structure. This suggests two dichotomous strategies for solving these kinds of geometry problems:

- Either embed or transform the original figure into a more highly structured figure in which the symmetries make the finishing-off algebra next to trivial, or

- Attempt to use brute-force algebra from the very beginning to solve the problem without embedding or transforming the original figure into a more highly symmetrized figure.

It's time now to throw in a bunch of solved problems as example of the use of the tools and methodology advocated above

Our eighth problem champions the Folded Butterfly tool

In Fig. 23 show that $\beta = \delta$.

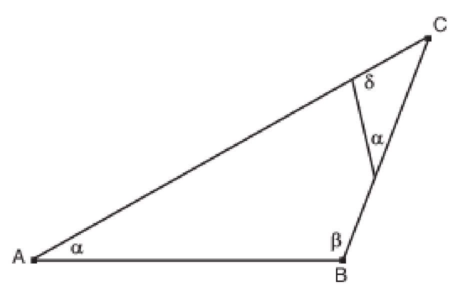

Figure 23. A folded butterfly.

From the Folded Butterfly Theorem, $\alpha + \delta = \beta + \alpha$, so $\beta = \delta$.

Our ninth problem uses the calculus of supplementary forms

A light ray enters a wedge of mirrors and travels the path indicated in Fig. 24. Show that $\beta = 2\alpha$.

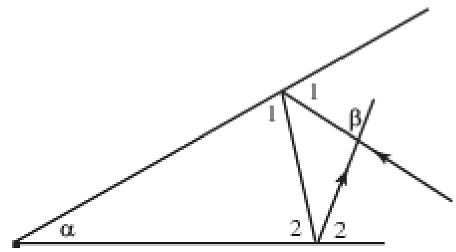

Figure 24. A light ray in a wedge.

From the Second Fundamental Theorem of the triangle we know that

$$\bar{a} = m\angle 1 + m\angle 2 \tag{31}$$

and using the Exterior Angle formula we know that

$$\begin{aligned} \beta &= \overline{2\angle 1} + \overline{2\angle 2} \\ &= 2\pi - (2m\angle 1 + 2m\angle 2) \\ &= 2[\pi - (m\angle 1 + m\angle 2)] \\ &= 2\alpha \end{aligned} \tag{32}$$

where we used (18f).

Our tenth problem also uses supplementary forms

In Fig. 25 is a triangle with two congruent angles and with a line segment bisecting the third angle. Show that $m\angle 3 = \pi/2$. We have two simple equations immediately from the figure: $m\angle 3 = m\angle 1 + m\angle 2$ and $m\,\overline{\angle 3} = m\angle 1 + m\angle 2$. From these we conclude that $m\angle 3 = m\,\overline{\angle 3}$, which, from (18i), leads us to conclude that $m\angle 3 = \pi/2$.

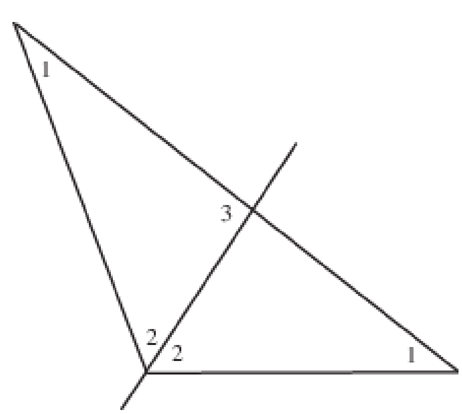

Figure 25.

Our eleventh problem is one of the important ones in H.S. geometry

Our problem here is to show that the angle bisectors of a triangle meet in a point. In Fig. 26 ΔABC has already had two of its interior angle bisected by $\overline{AF}$ at $\angle A$ and by $\overline{CE}$ at $\angle C$. These two line segments meet at G. Point D is the extension of line segment $\overline{BG}$ to $\overline{CA}$.

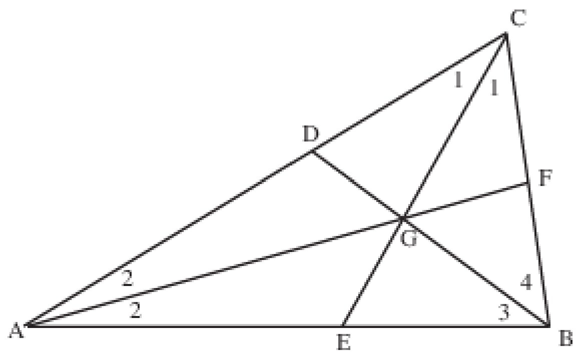

Figure 26.

From Problem Five we learned that the bisector of an interior angle of a triangle divides the opposite side of the triangle in the same ratio as the ratio of the other two sides of the triangle. Well, we've got two angle bisectors here and we could write down some equations relating lines segments on the big triangle ΔABC, but we have a much better approach by noticing that ΔCDB and ΔADB also both share this half-angle relation and they are joined so-to-speak at $\overline{BD}$, from which we can write:

$$\frac{DC}{BC} = \frac{DG}{GB} = \frac{DA}{BA} \tag{33}$$

Therefore

$$\frac{DC}{DA} = \frac{BC}{BA} \tag{34}$$

which is the condition that $\overline{BD}$ is the angle bisector at B.

Our twelfth problem uses the Butterfly symmetry again.

In Fig. 27 we depict two chords of a circle that intersect within the circle. Show that

$$PB \bullet AP = PC \bullet PD \tag{35}$$

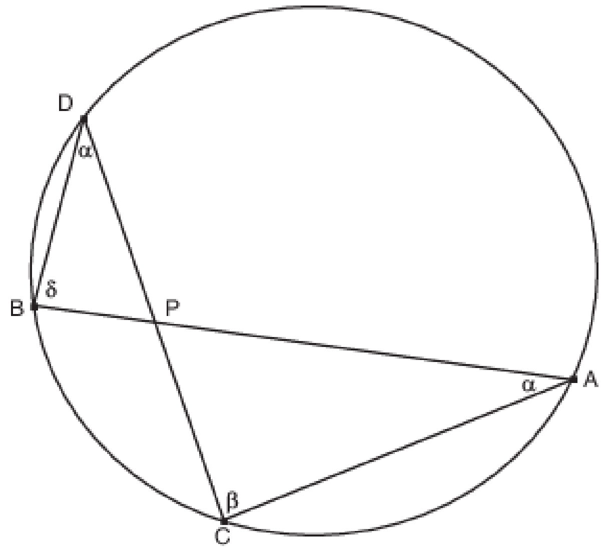

Figure 27. Two inscribed angles forming a butterfly.

The first thing to note is that (35) is just a "flattened" version of the proportion

$$\frac{PB}{PD} = \frac{PC}{AP} \tag{36}$$

OK, so let's do some "backward" or "would-be" reasoning at this point: Equation (36) would be true if $\Delta APC \sim \Delta DPB$. And that similarity relation would hold if all three pairs of corresponding angles were equal. Now we already know that $m\angle A = m\angle D$ because they both intercept the same minor arc. And we know therefore by the Butterfly Similarity Corollary that $\Delta APC \sim \Delta DPB$. And we are done.

Our thirteenth problem investigates another wedge — to use similar triangles or maps is the question.

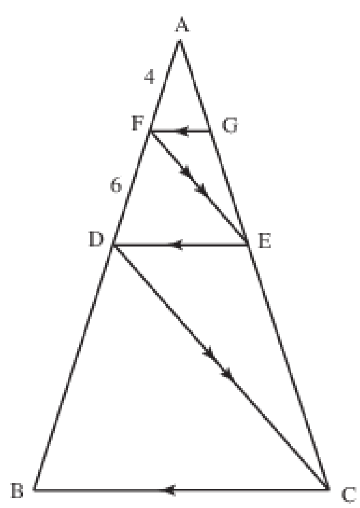

Figure 28.

In Fig. 28 is an isosceles triangle *ABC*. Parallel line segments are indicated by arrows in the standard way. $AF = 4$ and $FD = 6$. What is *DB*?

Let's solve this problem with maps! The horizontal parallel lines connect points of equal distance from point *A*. Let *M* be the map that takes *F* to *E*, then $M = (6+4)/4 = 5/2$. Map *M* connects points *F* and *C* through *E* and *D* (what's called a "pathway through the wedge" — just follow the arrows!), yielding $M^2(AF) = M^2(4) = (5/2)^2\, 4 = 25 = AC = AB$. Now, $DB = AB - AD = 15$.

Our fourteenth problem is a "challenging" variant on the last problem.

This problem would be challenging using similar triangles, but it is conceptually trivial using linear maps. In Fig. 29 we have another level of nested "similar" triangle. The point *H* has been added, with $AH = m_1$ and $HF = m_2$. What is *BD*?

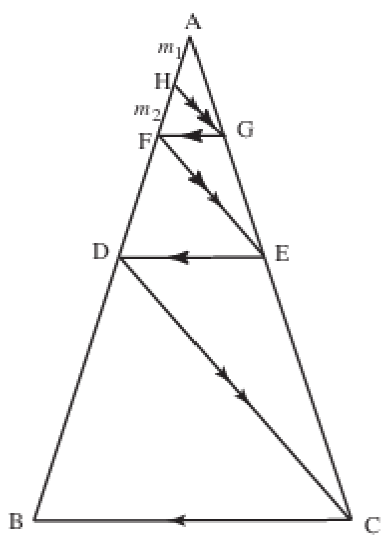

Figure 29. In terms of linear maps this is trivial!

Let *M* be the linear map that takes *H* to *G*, then $M(m_1) = m_1 + m_2$ and generally, $M^n(x) = (m_1 + m_2)^n\, x / m_1^n$.

So,

$$\begin{aligned} BD &= BA - DA = CA - EA \\ &= M^3(m_1) - M^2(m_1) \\ &= (m_1 + m_2)^3 / m_1^2 - (m_1 + m_2)^2 / m_1 = m_2(m_1 + m_2)^2 / m_1^2 \end{aligned} \qquad (37)$$

Done!

A nonstandard tool: Taking Inverses!

A wedge is a multiplication tool! We can use it to recursively multiply or to take multiplicative inverses. Consider Fig. 30a below. What we want is a sort of inverse in the ordinary sense of multiplication, that is $(AB)^{-1} = x/AB$, where, of course, AB is the length of segment $\overline{AB}$. When $x = 1$ then $(AB)^{-1}$ is the ordinary multiplicative inverse of AB.

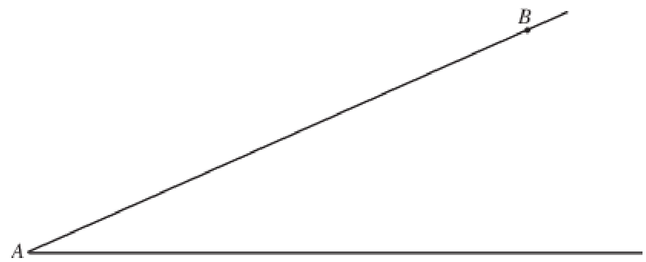

Fig. 30a. How to take the inverse of $\overline{AB}$?

But as the problem stands at this point, it is not possible to find such an inverse. We must first either define a unit length along the inclined line of the wedge or we must define some other point to act as a sort of "fulcrum" for taking inverses, and this is what we will do. To that end we can place any other point on line $\overline{AB}$, which we have done in Fig. 30b. For convenience we have put the point C between A and B, but this was not necessary. Now our search is to find the point D such that $AD = x/AB$.

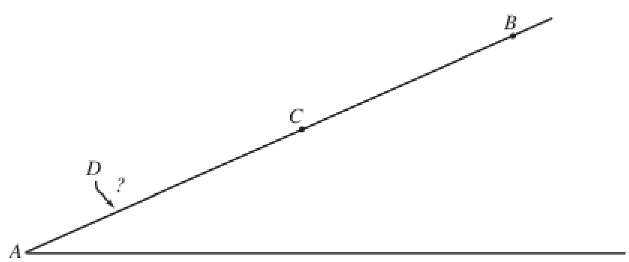

Fig. 30b. A fulcrum C has been chosen.

Now we use A as a centrum to circularly project points C and B on the inclined line to points C' and B', respectively, on the level line. This brings us to Fig. 30c.

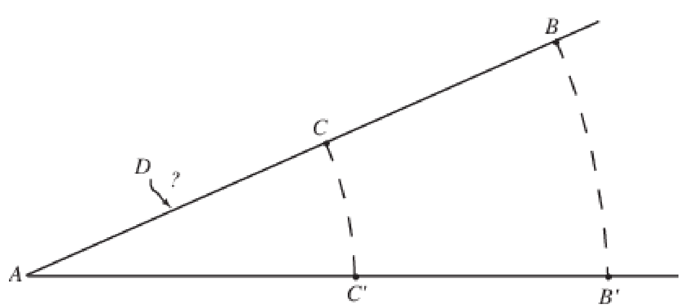

Fig. 30c. A is a centrum to circularly project points C and B on the inclined line to points C' and B', respectively, on the level line.

Our last step is to use parallel lines (you knew that they had to into the act somehow) to make similar triangles inside the wedge. This is depicted if Fig. 30d, which shows where the point *D* is located.

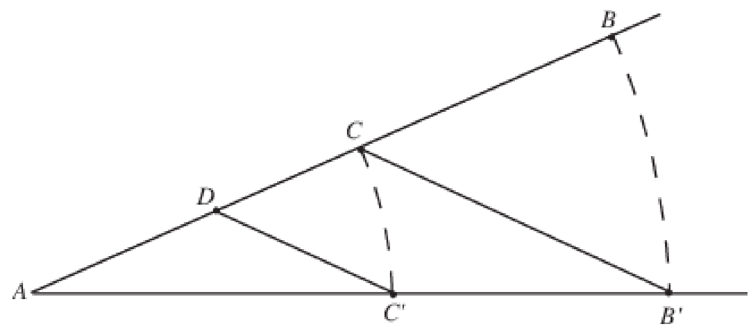

Fig. 30d. *D* is a generalized inverse to the point *B* with respect to point *C*.

Comparing the similar triangles, we get:

$$\frac{AD}{AC'} = \frac{AC}{AB'} \quad \Rightarrow \quad \frac{AD}{AC} = \frac{AC}{AB} \quad \Rightarrow \quad AD = \frac{(AC)^2}{AB}. \qquad (38)$$

Viewed from the perspective of a linear map from the inclined line to the level line, the map that takes the point C ($\overline{AC}$) to B'($\overline{AB'}$), has a generalized inverse that takes the point C'($\overline{AC'}$) to D ($\overline{AD}$). Either way you look at it, when AC is set equal to unity then $AD = 1/AB$.

Lemma on Interior Inverses:

Introduce into Fig. 30d three parallel line segments connecting *D*, *C*, and *B* with unique points on the line *AB′*.

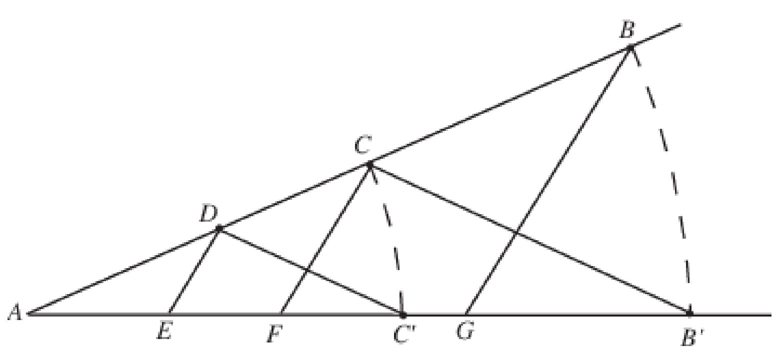

Figure 30e. Inverses in the wedge.

Also, let the line segments connecting points *A*, *D*, *C*, and *B* satisfy Equation (38). (See Fig. 30e.) Then

$$DE = \frac{(CF)^2}{BG} \tag{39}$$

or, in words: *DE* is the generalized inverse of *BG* with respect to *CF*.

Proof:

Since parallel lines cutting through a wedge create mutually similar triangle of some ratio of proportion say λ, then the following is true.

$$\lambda = \frac{AD}{DE} = \frac{AC}{CF} = \frac{AB}{BG} \tag{40}$$

Therefore

$$\frac{AD}{AC} = \frac{AC}{AB} \quad \Rightarrow \quad \frac{\lambda \cdot DE}{\lambda \cdot CF} = \frac{\lambda \cdot CF}{\lambda \cdot BG} \tag{41}$$

which, on canceling λ's yields

$$\frac{DE}{CF} = \frac{CF}{BG} \quad \Rightarrow \quad DE = \frac{(CF)^2}{BG}. \tag{42}$$

Our fifteenth problem returns to conjecture formulation

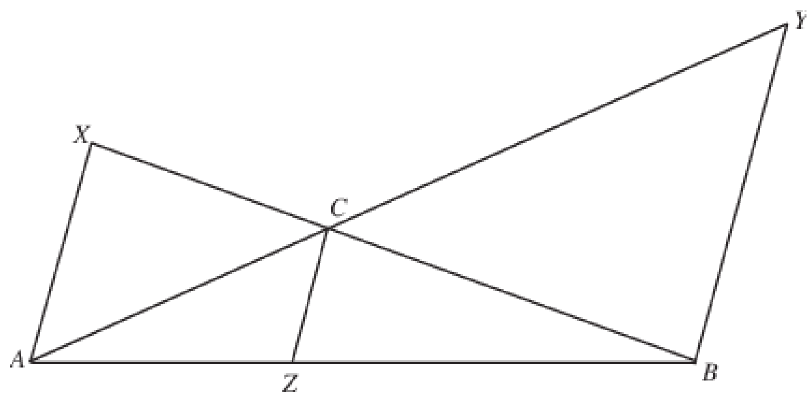

Figure 31a. How to use generalized inverses directly.

In Fig. 31 we have a situation that depicts what looks like two (different) wedges in roughly opposite orientations. The segments $\overline{AX}$, $\overline{CZ}$, and $\overline{BY}$ are mutually parallel. This problem can be found in Posamentier & Salkind on pages 10 and 74. They simply state an equation to be proved, but here we want to conjecture the equation first.

Therefore: conjecture by trial and error a constitutive relation among 1/*AX*, 1/*CZ*, and 1/*BY*. Note that the segments involved are inside the wedges *XBA* and *BAY*, which is new to our analysis. Hint: use the inverse-construction tools to find the inverses of *X* and *Y* with respect to fulcrum *C*, and then conjecture a relation among those generalized inverse inverses and *CZ*.

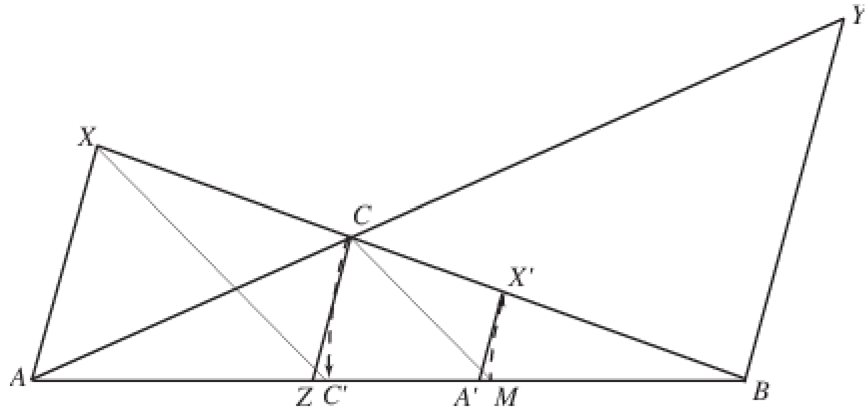

Figure 31b. First, we do wedge *XBA*.

Let's start with wedge *XBA*. In it we invert point *X* with respect to fulcrum *C* about centrum *B*. First, we circularly project *C* to *C′*. Then we connect *XC′* with a line segment. Then we draw a line parallel to *XC′* through *C*, forming the line segment (or line) *CM*. Now we circularly project *M* to *XB*, meeting it at a point we label *X′*. Finally, we draw a line through *X′* parallel to *XA*, meeting *AB* in the point *A′*. This brings us to the state found in Fig. 31b.

Now we do essentially the same process in wedge *BAY* to invert point *Y* with respect to fulcrum *C* about centrum *A*. The result is shown in Fig. 31c.

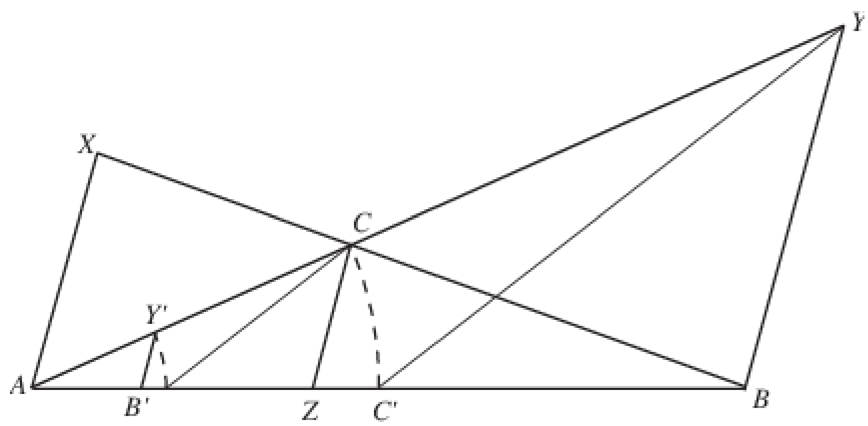

Figure 31c. Point *Y* has been inverted to point *Y′* in wedge *BAY*.

Now all we have to do is to put *A′X′* and *B′Y′* into ΔABC and make a conjecture. The result is displayed in Fig. 31d.

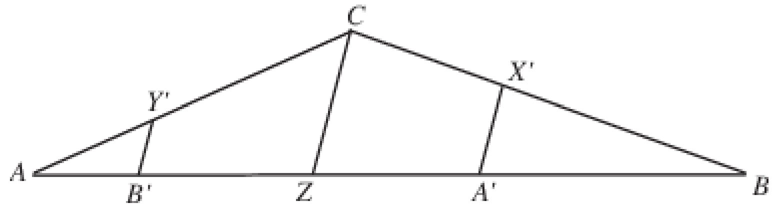

Figure 31d. What constitutive conjecture about this figure can we make?

Our last task is to make a constitutive conjecture that relates1/*AX*, 1/*CZ*, and 1/*BY*. First, we can make the plausible conjecture that

$$B'Y' + A'X' = CZ \tag{43}$$

Then we replace $B'Y'$ and $A'X'$ by the explicit generalized inverses in terms of *BY* and *AX* to get

$$\frac{(CZ)^2}{BY} + \frac{(CZ)^2}{AX} = CZ. \tag{44}$$

Finally, we divide (44) through by $(CZ)^2$ to get

$$\frac{1}{BY} + \frac{1}{AX} = \frac{1}{CZ}. \tag{45}$$

Posamentier & Salkind present a relatively short proof of (45) in their book. Since their proof is shorter than this conjecturing, why bother with conjecturing? The whole point of conjecturing, besides the fact that it is more fun than just proving relationships, is that it gets us out of the artificial world of formal problems and gets us into a realm much more

We shall present our own proof of this relation after building better tools for pattern matching.

Theorem: Parallel-Winged Butterfly.

We have already seen he power of using the Butterfly for pattern-matching, but now we will investigate a particular type of Butterfly that has a great deal more symmetry than an arbitrary one. Consider the case of a Butterfly construction where the edges of the "wings" of the Butterfly are parallel to each other, as depicted in Fig. 32.

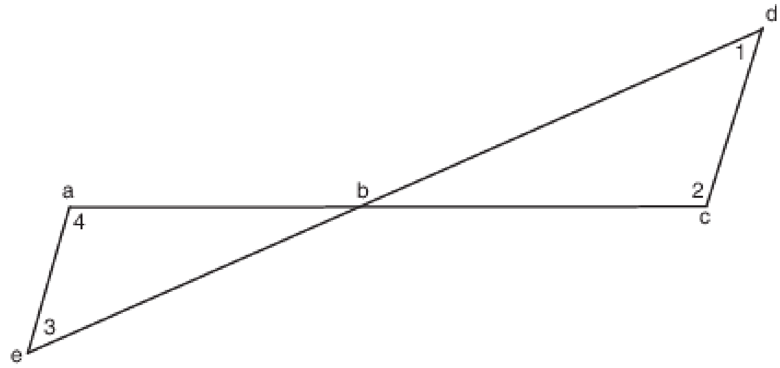

Figure 32. The Parallel-Winged Butterfly forms two similar triangles.

In Fig. 32 $\Delta bae \sim \Delta bcd$. From which we get that

$$\frac{ab}{eb} = \frac{cb}{db}, \tag{46a}$$

$$\frac{ab}{eb} = \frac{cb}{db}. \tag{46b}$$

Now we are ready to prove the relation in Equation (45). Let's begin with a suitable constitutive equation, based on Fig. 31a,

$$YA = YC + CA. \tag{47}$$

Clearly we should divide (47) through by the quantity *YB*, first because we want a $YB = BY$ in the denominator of one of the terms to begin the process of *molding* our present equation into the ShowThat equation, and second because we need some artifice to get rid of *YA*. After the division we get

$$\frac{YA}{BY} = \frac{YC}{BY} + \frac{CA}{BY}. \tag{48}$$

From wedge *YAB* we get that $YA/BY = CA/CZ$. And from butterfly *YCAXB* we get that $YC/BY = CA/AX$. On substituting these two results into (48) we get

$$\frac{CA}{CZ} = \frac{CA}{AX} + \frac{CA}{BY}. \tag{49}$$

And finally, by dividing (49) by *CA* we get that

$$\frac{1}{CZ} = \frac{1}{AX} + \frac{1}{BY}. \tag{50}$$

Henceforth we shall refer to this result as the **Two-Peaks Theorem**.

Is there a method to our madness?

Let’s take a moment to reflect on the progress we have made so far. We started with the simple but fundamental symmetry that when parallel lines are cut by a transversal the corresponding angles generated are congruent. Then we formed the wedge and utilized the symmetries of similar triangles to get generally powerful results. Next, we identified the symmetries of the Butterfly configuration, and then the stronger but less frequent needed Parallel-Winged Butterfly. Lastly, we identified the Two-Peaks Theorem. The progression listed here is obviously from the simple to the complex. The simpler ones are the more general but the least efficient. The more complex ones are the more powerful but less general tools for pattern-matching.

You see, our search for a proof of some relation that bears on a particular figure is like a cryptologist’s search for sets of keys or patterns that can be made or matched against to decode an encrypted message. Well, a problem in geometry is like an encrypted message. If we can find the right pattern to match the figure against, we can, together with a good set of heuristics, quickly find the appropriate relations to solve the problem. Thus, the problem-solving strategy proposed here is to develop a good set of heuristic and a large bag of pattern-matching tools to help reveal the symmetry, structure, and algebraic relations in any given figure.

To attempt a convincing demonstration of both our pattern-matching tools and our general methodology we will now re-solve the first problem by using the strong symmetries of the Parallel-Winged Butterfly tool as a pattern-matching tool. Refer once more to Fig. 1. Note the two Parallel-Winged Butterflies *DARCT* and *DCRAS*. From the first we get that

$$\frac{RD}{RT} = \frac{RA}{RC} \tag{51a}$$

and from the second we get that

$$\frac{RA}{RC} = \frac{RS}{RD}. \tag{51b}$$

Then, on eliminating RA/RC between these two equations, we get Equation (1) for almost no effort at all. If we don’t build-up evermore complex pattern-matching tools to work with then we must always begin with the most primitive tools, which is obviously inefficient.

REFERENCES

A. Posamentier & C. Salkind, [1970, 1988], 1996, *Challenging Problems in Geometry*, Dover Publ., Mineola, N.Y.