Semidirect Products by Martin Baker

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Abstract

In this paper I present my read-along notes to the YouTube video made by Martin L. Baker on semidirect products and extensions. The blame for all errors (if any) in this paper is the fault of the author.

1 Where found on YouTube

Martin Baker's YouTube video on which I have made these companion notes can be found at

https://www.youtube.com/watch?v=tFmbJEaEx3Q

The exact title of the video is "Group theory: Semidirect products, extensions #2." So, we have some interesting stuff presented to us.

I should point out that this video is the second in a series of videos uploaded by Baker on inner semidirect products, and that this video starts off with a quick review of the previouis video. I will not show this review but rather refer the reader to the previous video if he or she has trouble following this presentation.

Note: All sequences shown in this paper will be exact.

2 The Splitting Lemma on Abelian Categories

If the reader is not familiar with category theory, that's okay. In that case just think of the objects of interest to be abelian groups.

We begin with a sequence of abelian objects:

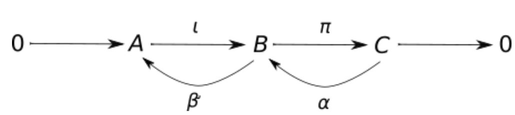

Figure 1. The Splitting Lemma over an abelian category. β associates to a 'left split', α to a 'right split'. The terminal objects of 0 could alternatively be set at as 1 or e, depending on how the author wants to denote the identity element of the objects.

In Figure 1, we find a diagram of a sequence of abelian objects. The map i is an injective homomorphism from A to B. The map π is a surjective homomorphism from B to C.

We will state without proof the following three equivalences:

- Left Split: There exist $\beta : B \to A$ s.t. $\beta \circ \iota = \text{id}_A$.
- Right Split: There exist $\alpha : C \to B$ s.t. $\pi \circ \alpha = \text{id}_C$.
- Direct Sum Property (same as the direct product for finite sets): $B \cong B \oplus C$.

3 What is a short exact sequence?

A short exact sequence is a sequence of groups (in our case) as in the figure below:

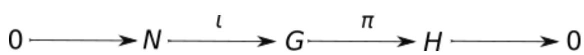

Figure 2. The Short Exact Sequence. Each arrow represents a homorphism.
The ι homomorphism in injective. The π homorphism is surjective.
The image of N is a normal subgroup of G. Lastly, $G/N \cong H$.

A sequence is said to be *exact* if it is exact at every object in the sequence (in our case, these objects are all groups).

Definition: A sequence is said to be *exact* at every object if the image from the immediate left is equal to the kernel to the immediate right. In this general case, a sequence is said to be exact at an object G_i if

$$\text{im}(G_{i-1}) = \ker(G_i)\,. \tag{1}$$

In the special case shown in Fig. 2, the sequence is exact at G if

$$\text{im}(\iota) = \ker(\pi)\,. \tag{2}$$

The first map from 0 to N is a homomorphism, so it must send the identity to the identity of N. And the exactness at this node N means that the identity of N is the kernel of π. This fact forces the ι homomorphism to be injective. Therefore,, the image of N under ι is a subgroup of G isomorphic to N, which we'll call $\overline{N}$. On top of that, we are told that this subgroup $\overline{N}$ is the kernel of the homomorphism π. But it's a well-know theorem of group theory that the kernel of a homomorphiosm from group A to group B is a normal subgroup of group A. Thus, $\overline{N}$ is normal in G.

From this point on, we will drop the overbar on subgroup N and just refer to it as N, and we have that

$$N \lhd G\,. \tag{3}$$

So, were it not for the First Isomorphism Theorem of Group theory, there would probably not be a use for the 'short exact sequence'. There is more utiitlity in this sequence of five nodes connect by four arrows than just to schematize the First Isomorphism Theorem: It is used to study the extension of one group by another and for the mashing together of two groups into a third group.

Next, we notice that every element of the group H gets mapped to the identity element 0. By definition, that means that every element of H is in the kernel of this last homomorphism. But we are told that every element of π is in the kernel of the terminal homomorphism. This forces us to conclude that the homomorphism π is surjective. And thus we can conclude that

$$G/N \cong H\,. \tag{4}$$

Next, we refer to Fig. 2 and add to it a left split homomorphism α such that

$$\pi \circ \alpha = \text{id}_H\,, \tag{5}$$

which brings us to Fig. 3.

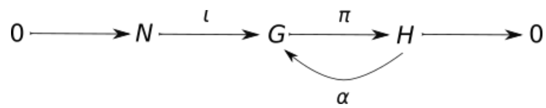

Figure 3. The Short Exact Sequence with a right split, in conformance with Eq. (5)

Definition: The group G is said to be an *extension* of H by K if G contains H as a normal subgroup, and $G/H \cong K$.

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4 Theorem on a split extension of N by H.

Theorem: The group G is an inner semidirect product of N and H if and only if G is a split extension of N by H.

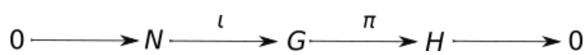

Figure 4. We will use this sequence to construct a right split from H to G. The ι homomorphism in injective. We need to show that π is surjective.

I'm inserting here a clarification statement for myself, if for no one else. We have two cases to prove.

($\Rightarrow$) Case 1) The short, exact sequence exists and G is a semidirect product of N and H, from which we conclude that G is split extension of N by H (that is, there exists an α such that $\pi \circ \alpha = \mathrm{id}_H$).

($\Leftarrow$) Case 2) The short, exact sequence exists and G is split extension of N by H, then G is a semidirect product of N and H.

Proof: ($\Rightarrow$)

Suppose G is the inner semidirect product of N and H, with $N \lhd G$. To show that the sequence is right split, we begin by defining the map $\pi : G \to H$ by $\pi(a) = h$ where $a = nh$, $\forall n \in N$ and $\forall h \in H$. (The factorization of a into nh is unique.)

Now, that π is a homorphism follows:

$$\begin{aligned}\pi(a_1 a_2) &= \pi(n_1 h_1 n_2 h_2) \\ &= \pi(n_1 h_1 n_2 h_1^{-1} h_1 h_2) \quad \text{(by virtual emplacement)} \\ &= \pi(n_1 n_2' h_1 h_2)) \quad \text{(for some } n_2' \in N) \\ &= h_1 h_2 \\ &= \pi(a_1)\pi(a_2)\,. \end{aligned} \tag{6}$$

Given that the sequence is exact, then π is surjective and $G/N \cong H$.

(I'm currently using the symbol 'ɥ' to mean 'for some'. This is a nonstandard usage. In fact, there is no commonly accepted symbol to represent 'for some'. On technical grounds, one could use the symbol '∃' but nobody uses that that I have ever seen. I'm also considering the use of the symbols ¶ and ♮. as possibles.)

Note: $ker\,\pi = N$, $\mathrm{im}\,\pi = H$. Hence, G is the extension of N by H.

Define $\alpha : H \to G$ by $\alpha(h) = h$.

$$\pi \circ \alpha(h) = \pi(\alpha(h)) = \pi(h) = h\,. \tag{7}$$

Therefore

$$\pi \circ \alpha = \mathrm{id}_H\,. \tag{8}$$

Going the other way ($\Leftarrow$):

Suppose G is a split extension of N by H.

Note: $N \lhd G$ and $G/N \cong H$. Now, since the sequence is split:

$$\pi : G \to H \quad \text{and} \quad \alpha : H \to G \tag{9}$$

and

$$\pi \circ \alpha = \mathrm{id}_H\,. \tag{10}$$

Define $\bar{H} = \mathrm{im}(\alpha)$ with $\bar{H} < G$. Question: Is $N \cap \bar{H}$ trivial?

Let $a \in N \cap \bar{H}$ then

$$a = \alpha(h) \;\; ɥ h \in H\,. \tag{11}$$

Hence, $\pi(a) = e$. So, $\pi(\alpha(h)) = e$, Therefore

$$\pi(h) = e \Rightarrow h = e.\,. \tag{12}$$

Hence

$$a = \alpha(e) = e \Rightarrow h = e\,. \tag{13}$$

Therefore,

$$N \cap \bar{H} = \{e\}\,, \tag{14}$$

making the intersection trivial.

My proof that $N \cap \bar{H} = \{e\}$.

For $a \in H$, $\alpha(a) = nh \in \bar{H}$ for some $n \in N$ and for some $h \in \bar{H}$. But if $nh \in \bar{H}$ then $n \in \bar{H}$. In which case

$$\pi(n) = \pi(ne) = e\,. \tag{15}$$

If $n \neq e$, we have a problem, because $\pi(\alpha(a)) = a$ for all $a \in H$. Therefore, this $n = e$ and the intersection is trivial.

Now, take $a \in G$ then

$$a = a\alpha(\pi(a))^{-1}\alpha(\pi(a)) = e\,. \tag{16}$$

So,

$$\pi(a) = \pi(a\alpha(\pi(a)^{-1})) = \pi(a)\pi(a)^{-1} = e \Rightarrow h = e\,. \tag{17}$$

Hence, $n \in \overline{N}$.

Given: G is a split extension of N by H.

a) $N \lhd G$ (by the inclusion homomorphism $N \hookrightarrow G$).

b) So, we need to use α to derive $\bar{H} < G$ such that $\bar{H} \cong H$ (to be provided by the homomorphism α).

To show that

$$G \cong N \rtimes H\,, \tag{18}$$