Proof that $\partial_\mu J^\mu = 0$ in a QFT Problem Using Isotropic Spinors

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Abstract

This paper demonstrates the use of isotropic spinors (2-d complex vectors)[1] to prove the fundamental result that $\partial_\mu J^\mu = 0$, where J^μ is the current vector for a problem in quantum field theory.

1 Introduction

This paper is a further application of those given in my paper "2nd-order Linear Differential Equations with Isotropic Spinors." Here, we prove the result that given the equation of motion of two particles of the same mass

$$(\partial^2 + m^2)\phi_\alpha + \lambda \boldsymbol{\phi}^2 \phi_\alpha = 0 \quad (\alpha = 1, 2)\,, \tag{1}$$

where ϕ_1 and ϕ_2 are the two solutions to (1), and $\boldsymbol{\phi}^2 \equiv \phi_1^2 + \phi_2^2$ [2] and $\partial^2 \equiv \partial_\mu \partial^\mu = \partial^\mu \partial_\mu$ $(\mu = 0, 1, 2, 3)$; then[3]

$$\partial_\mu J^\mu = 0\,, \tag{2}$$

where

$$J^\mu \equiv i(\phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1)\,. \tag{3}$$

I saw this problem while watching a YouTube video by Anthony Zee called "Quantum Field Theory," (Lecture 3 of 4) from 2013. The version he used in the video is (1), though, in his textbook, on page 77, it appears in a slightly different form, though that won't matter for our purposes here.

Zee suggested that if one would only stare long enough at (1) and (3), then one should be able to somehow see one's way through to prove Equation (2). The rest of this paper is the result of my staring at and ruminating on these equations and employing isotropic spinors to facilitate the algebra involved.

[1] These 'spinors' are not associated with particle spin.
[2] This is by my assumption.
[3] In this paper, the Einstein summation convention will always be in effect.

2 Formalism

To facilitate our computations, we introduce here a symplectic scalar product, we take the **symplectic matrix** $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,[4] and get, with arbitrary 2×1 matrix/vector B^α (where $\alpha = 1, 2$),

$$B_\alpha \equiv (JB^\alpha)^{\mathrm{T}} = B^{\alpha \mathrm{T}} J^{\mathrm{T}} , \tag{4}$$

to get

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (B^2, -B^1) . \tag{5}$$

So our symplectic scalar product acts like the antisymmetric cross product of two vectors.[5] Hence, for any A_α,

$$A_\alpha A^\alpha = 0 \quad \text{and} \quad A^\alpha A_\alpha = 0 , \tag{6}$$

where we have summed on the index α.

Definition: A nonzero vector/spinor whose 'square' is zero is said to be **isotropic**. In the case of the 'symplectic square', getting zero is by design.

3 Lemmas:

Lemma 1 On multiplying (1) through by ϕ^α and using (6), we get that

$$\phi^\alpha \partial^2 \phi_\alpha = 0 . \tag{7}$$

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Lemma 2

$$(\partial_\mu \phi^\alpha)(\partial^\mu \phi_\alpha) = 0 . \tag{8}$$

So, on expanding the sum and them applying (5), we have

$$\begin{aligned} (\partial_\mu \phi^\alpha)(\partial^\mu \phi_\alpha) &= (\partial_\mu \phi^1)(\partial^\mu \phi_1) + (\partial_\mu \phi^2)(\partial^\mu \phi_2) \\ &= (-\partial_\mu \phi_2)(\partial^\mu \phi_1) + (\partial_\mu \phi_1)(\partial^\mu \phi_2) \\ &= -(\partial^\mu \phi_2)(\partial_\mu \phi_1) + (\partial_\mu \phi_1)(\partial^\mu \phi_2) \\ &= 0 . \end{aligned}$$

Going from step 2 to step 3 is justified in the Appendix.

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[4] Please don't confuse this 2×2 matrix represented by J from symplectic theory with the 4×1 vector J^μ from special relativity.

[5] It's easy to show that $A_\alpha B^\alpha = -B_\alpha A^\alpha$.

4 Proof of main result:

Definition:

$$W^\mu \equiv \phi^\alpha \partial^\mu \phi_\alpha \qquad (\mu = 0, 1, 2, 3)\,. \tag{9}$$

Technically, we don't need this definition, but I think it helps us make a bridge between this physics quantum stuff and the 'Wronskian' of the theory of second-order linear differential equations.[6]

Definition:

$$J^\mu \equiv iW^\mu\,, \tag{10}$$

which is consistent with (3).

So, in order to show that $\partial_\mu J^\mu = 0$, it is sufficient to show that

$$\partial_\mu W^\mu = 0\,. \tag{11}$$

After applying the product rule, we then apply (7) and then (8), we get

$$\begin{aligned} \partial_\mu W^\mu &= \partial_\mu(\phi^\alpha \partial^\mu \phi_\alpha) \\ &= (\partial_\mu \phi^\alpha)(\partial^\mu \phi_\alpha) + \phi^\alpha \partial_\mu \partial^\mu \phi_\alpha \\ &= (\partial_\mu \phi^\alpha)(\partial^\mu \phi_\alpha) \\ &= 0\,. \end{aligned} \tag{12}$$

Therefore, we have established that

$$\partial_\mu J^\mu = 0\,. \tag{13}$$

5 Conclusion:

We have gained two advantages by using isotropic spinors in this problem. First, by reason of the isotropic nature of the spinors, we can, for example, go from (1) to (7) is one easy step (in fact, by inspection).

Second, by indexing the solutions of (1) into a 'spinor', we are able to write more compact expressions, such as

$$J^\mu \equiv i\phi^\alpha \partial^\mu \phi_\alpha \qquad (\mu = 0, 1, 2, 3)\,, \tag{14}$$

rather than (3).

[6]Our W^μ is not exactly a Wronskian, because the Wronskian of second-order linear differential equations is a function of a single independent variable.

6 Appendix:

For function f, g,

$$(\partial_\mu f)(\partial^\mu g) = (\partial^\mu f)(\partial_\mu g)\,. \tag{15}$$

Proof: (Note: $\eta_{\beta\mu}$ is the spacetime metric tensor.)

$$\begin{aligned}(\partial_\mu f)(\partial^\mu g) &= \delta^\nu_\mu(\partial_\nu f)(\partial^\mu g) \\ &= \eta^{\beta\nu}\eta_{\beta\mu}(\partial_\nu f)(\partial^\mu g) \\ &= (\eta^{\beta\nu}\partial_\nu f)(\eta_{\beta\mu}\partial^\mu g) \\ &= (\partial^\beta f)(\partial_\beta g) \\ &= (\partial^\mu f)(\partial_\mu g)\,,\end{aligned} \tag{16}$$

where we have replaced the dummy index β by the dummy index μ.