

Proving Pappus's Theorem with Isotropic Spinors

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August 20, 2020

Abstract

Pappus's Theorem is proved using isotropic spinors. The proof is not the shortest available, nor is it the most elegant, but that's not really the point. The point is for me to demonstrate the fullest extent of applicability of isotropic spinors to various areas of mathematics, and projective geometry is one of those areas.

Introduction:

In the figure below, L_1 and L_2 are lines in a plane. Point X is the intersection of line segments \overline{AB} and \overline{DE} , point Y is the intersection of line segments \overline{DC} and \overline{AF} , and point Z is the intersection of line segments \overline{BC} and \overline{EF} . The Theorem of Pappus states that the points X , Y , and Z are collinear. For this, my first attempt at proving Pappus's Theorem with spinors, I will restrict my application to the case where the lines in the plane are *not* parallel and thus must intersect.

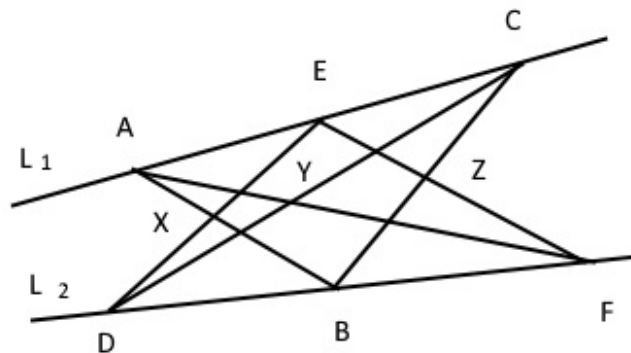


Figure 1. The Theorem of Pappus.

Proof:

We begin by taking stock of the given information, or rather, given constraints. First, we have that points on L_1 , A , E , and C are collinear. We have a similar constraint for points on L_2 . That's two constraints, each requiring some

algebraic representation. We also have six line segments, each containing three point. For example, we have that points D , X , and E are collinear on line segment \overline{DXE} . That's a total of eight constraints that have to be prepared in the formalism of the spinors before we seek a means to combine them to prove that points X , Y , and Z are collinear.

We will treat the points in this plane as the end points of vectors whose origin is anywhere you like for the time being, except that it cannot lie on any of the displayed points. The symplectic inner product is like taking the cross product of two vectors: If their product is zero, the two points are collinear with the origin. If from three points we take the differences of two pairs of them, and then take the inner product of those differences and get zero, the three points are collinear.

For example, let x , y , z be any three distinct points in the plane. Then $x - y$ and $x - z$ are two distinct vectors in the plane. Now, x , y , z are collinear if and only if the vectors $x - y$ and $x - z$ are scalar multiples of each other. In conventional vector analysis, one could show this by proving that, for some scalar μ , $(x - y) = \mu(x - z)$, or by showing that their cross product is zero: $(x - y) \times (x - z) = 0$. However, to do this, one must embed the plane into three space. But our use of symplectic inner product allows us to have all the power of cross products without this awkward contrivance.

Using isotropic spinors, the 'cross product' of two vectors (spinors) A and B , say, is $A^\alpha B_\alpha$.¹ To show that the three points A , E , and C are collinear, we could show any of

$$(A^\alpha - E^\alpha) = \mu(A^\alpha - C^\alpha), \quad (1a)$$

$$(A^\alpha - E^\alpha)(A_\alpha - C_\alpha) = 0, \quad (1b)$$

$$A^\alpha E_\alpha + E^\alpha C_\alpha + C^\alpha A_\alpha = 0. \quad (1c)$$

In my effort to prove this theorem, for the longest time I tried to find an elegant solution by placing all the constraints in the form of (1b), but I could never find a way to get as my output that

$$(X^\alpha - Y^\alpha)(X_\alpha - Z_\alpha) = 0. \quad (2)$$

Finally, the pathway to a proof came when I decided to solve for X , Y , and Z in terms of the 'line' points (i.e., the points lying on the two lines), by use of scalar multiples, as in (1a). For example, to write X (or X^α) in terms of the line points, we could do this in a number of ways, such as,

$$X^\alpha = D^\alpha + \gamma(E^\alpha - D^\alpha). \quad (3)$$

which, interpreted, means, go to point D^α and then travel along the vector $(E^\alpha - D^\alpha)$ γ units. However, we can't make good use of this approach unless we can solve for these scalar factors in terms of the line points, so let's do that

¹This is equivalent to the z-component of $\mathbf{B} \times \mathbf{A}$, and $A_\alpha B^\alpha$ is equivalent to the z-component of $\mathbf{A} \times \mathbf{B}$.

now. To accomplish this, we need the other way to get to X^α , which is to start at one of the endpoints of line segment \overline{BXA} . I chose to start at B^α :

$$X^\alpha = B^\alpha + g(A^\alpha - B^\alpha). \quad (4)$$

Setting these two X^α 's equal, gives

$$D^\alpha + \gamma(E^\alpha - D^\alpha) = B^\alpha + g(A^\alpha - B^\alpha). \quad (5)$$

Subtract D^α from both sides and we get,

$$\gamma(E^\alpha - D^\alpha) = (B^\alpha - D^\alpha) + g(A^\alpha - B^\alpha). \quad (6)$$

Now multiply through by $(A_\alpha - B_\alpha)$ – and sum, of course! We get

$$\gamma(E^\alpha - D^\alpha)(A_\alpha - B_\alpha) = (B^\alpha - D^\alpha)(A_\alpha - B_\alpha). \quad (7)$$

It's important to note that this last equation is completely scalarized, so we can easily solve for γ :

$$\gamma = \frac{(B^\alpha - D^\alpha)(A_\alpha - B_\alpha)}{(E^\alpha - D^\alpha)(A_\alpha - B_\alpha)}. \quad (8)$$

Proceeding similarly for points Y^α and Z^α , we get

$$Y^\alpha = F^\alpha + \tau(A^\alpha - F^\alpha), \quad (9)$$

where

$$\tau = \frac{(D^\alpha - F^\alpha)(C_\alpha - D_\alpha)}{(A^\alpha - F^\alpha)(C_\alpha - D_\alpha)}, \quad (10)$$

and

$$Z^\alpha = B^\alpha + \mu(C^\alpha - B^\alpha), \quad (11)$$

where

$$\mu = \frac{(E^\alpha - B^\alpha)(E_\alpha - F_\alpha)}{(C^\alpha - B^\alpha)(E_\alpha - F_\alpha)}. \quad (12)$$

From here the way is straightforward, but tedious. I introduced the variable Φ as a convenience for a later step:

$$\Phi = (X^\alpha - Y^\alpha)(X_\alpha - Z_\alpha), \quad (13)$$

and I showed that, after inserting X^α , Y^α , and Z^α from (3), (9), and (11), respectively, I got zero!

Now, as we multiply out these spinors in Φ , we'll get two kinds of products of points: Either both points will be from the same line or from different lines, the former I'll refer to as *monads* and the latter as *dyads*. And it's at this point I make a much needed algebraic simplification. Since I'm free to choose to put the origin (almost) anywhere I choose, I choose to put it at the intersection of L_1 and L_2 (the *nexus gauge*), that way, all the monads are identically zero! This has the added benefit of utilizing both collinear constraints first of A , E , and

C being on L_1 , and second of D , B , and F being on L_2 . Examples of monad products being zero are $A^\alpha E_\alpha = 0$ and $D^\alpha F_\alpha = 0$. But no dyad, such as $B^\alpha E_\alpha$, is zero.

So, after substituting into (13), we get

$$\begin{aligned} \Phi = & [(D^\alpha - F^\alpha) + \gamma(E^\alpha - D^\alpha) - \tau(A^\alpha - F^\alpha)] \\ & \times [(D_\alpha - B_\alpha) + \gamma(E_\alpha - D_\alpha) - \mu(C_\alpha - B_\alpha)]. \end{aligned} \quad (14)$$

After a bit more work, we get

$$\begin{aligned} \Phi = & \gamma(-F^\alpha E_\alpha - E^\alpha B_\alpha) && \text{L1} \\ & + \mu(F^\alpha C_\alpha - D^\alpha C_\alpha) && \text{L2} \\ & + \tau(A^\alpha B_\alpha - A^\alpha D_\alpha) && \text{L3} \\ & + \mu\gamma(E^\alpha B_\alpha + D^\alpha C_\alpha) && \text{L4} \\ & + \gamma\tau(A^\alpha D_\alpha + F^\alpha E_\alpha) && \text{L5} \\ & + \tau\mu(-A^\alpha B_\alpha - F^\alpha C_\alpha) && \text{L6} \end{aligned} \quad (15)$$

It's important to note that we've reached an important milestone in the progression of our proof, for we have now fully utilized all the eight given constraints, and no further appeal to them is required. It's just high-school algebra from here on. To further simplify matters, I arbitrarily assigned to each dyad a distinct letter name:²

$$\begin{aligned} B^\alpha E_\alpha = a, & & D^\alpha C_\alpha = d, \\ F^\alpha E_\alpha = b, & & D^\alpha A_\alpha = e, \\ B^\alpha A_\alpha = c, & & C^\alpha F_\alpha = f. \end{aligned} \quad (16)$$

With these, our scale factors become

$$\tau = \frac{f+d}{e+f}, \quad \gamma = \frac{e-c}{e-a}, \quad \mu = \frac{a-b}{f+a}. \quad (17)$$

Using these symbols, (15) becomes, after we clear out the denominators,

$$\begin{aligned} (e-a)(f+a)(e+f)\Phi = & (e-c)(a-b)(f+a)(e+f) && \text{L1} \\ & + (a-b)(-f-d)(e-a)(e+f) && \text{L2} \\ & + (f+d)(e-c)(e-a)(f+a) && \text{L3} \\ & + (a-b)(e-c)(d-a)(e+f) && \text{L4} \\ & + (e-c)(f+d)(b-e)(f+a) && \text{L5} \\ & + (f+d)(a-b)(c+f)(e-a). && \text{L6} \end{aligned} \quad (18)$$

The sum of the odd-numbered lines simplifies to

$$L_1 + L_3 + L_5 = (a-b)(c-e)(d-e)(a+f), \quad (19)$$

²Note: $E^\alpha B_\alpha = -a$, etc.

and the sum of the even-numbered lines simplifies to

$$L_2 + L_4 + L_6 = (a - b)(c - e)(-d + e)(a + f). \quad (20)$$

Therefore the sum of all six lines is zero, implying that $\Phi = 0$, and the theorem is proved.

Conclusion

The isotropic nature of the spinors we defined has afforded us a convenient means to deal with questions of collinearity of points and the parallelism of multiple lines. I believe that the proof contained here is well within the reach of the high school student who is already familiar with vector algebra. Only a little preparation to get the student familiar with the symplectic inner product should be enough foundation.

Finally, it should be a simple matter to translate the proof given here into the geometric algebra of the plane [1].

1 Appendix: Brief Summary of Isotropic Spinors

We start with any two vectors \mathbf{A} and \mathbf{B} in 2-space and represent them in matrix form as $\begin{pmatrix} A^1 \\ A^2 \end{pmatrix}$ and $\begin{pmatrix} B^1 \\ B^2 \end{pmatrix}$, respectively. These vectors also have a convenient component, or indicial, form as A^α and B^α , respectively, where α takes on values 1,2.

Now, you're probably familiar with the Euclidean inner product of two vectors, but we aren't going to use that inner product. (If we're going to capture the information in a cross product as a 'scalar product', this makes sense to use some other notion of inner product.) But for comparison's sake, we'll show what the Euclidean inner product of two vectors B^α, A^α looks like:

$$B_\alpha A^\alpha \equiv \sum_\alpha B_\alpha A^\alpha = B_1 A^1 + B_2 A^2. \quad (21)$$

In other words, to scalar-multiply two column vectors together, we will have to convert one of them to a row vector by this procedure: $B_\alpha = [JB^\alpha]^t = (B^\alpha)^t J^t$, where J is a 2×2 matrix used to lower the index of a vector. For the Euclidean case, J is the 2×2 identity matrix, and

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (B^1, B^2), \quad (22)$$

with the resulting familiar Euclidean square

$$B_\alpha B^\alpha = (B^1, B^2) \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} = (B^1)^2 + (B^2)^2. \quad (23)$$

However, for our symplectic inner product, we take the symplectic matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and get³

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (-B^2, B^1). \quad (24)$$

where $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J^t$ is the transpose of the symplectic matrix.

Now we're ready to take the symplectic 'inner' product of vectors B^α and A^α :

$$B_\alpha A^\alpha = (B_1, B_2) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = (-B^2, B^1) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = -B^2 A^1 + B^1 A^2 \quad (25)$$

And, on taking this inner product in the opposite order, we get

$$A_\alpha B^\alpha = -A^2 B^1 + A^1 B^2 = -(-B^2 A^1 + B^1 A^2) \quad (26)$$

Comparing (25) to (26), we see that they are negatives of each other

$$A_\alpha B^\alpha = -B_\alpha A^\alpha \quad (27)$$

So our symplectic inner product acts like the antisymmetric cross product of two vectors.

Definition: So far the components of our two-component vectors have been real numbers or real-valued functions. But now we let them be complex-valued, in which case we refer to them as **spinors**. Spinors were introduced to physics by Paul Ehrenfest and Wolfgang Pauli in the 1920s to deal with the then novel notion of electron spin. We will not need that particular interpretation of them. Our use of them here is quite formal, but convenient, as we shall see.

Important Lemma

Let A^α and B^α be two spinors. Then their symplectic 'inner' product is

$$A_\alpha B^\alpha = \det \begin{bmatrix} A^1 & A^2 \\ B^1 & B^2 \end{bmatrix} = A^1 B^2 - B^1 A^2. \quad (28)$$

But what happens if we use the same vector in this scalar product? Let's try it. $A_\alpha A^\alpha = -A^2 A^1 + A^1 A^2 = 0$. And this is true whatever the components of A^α are! This is certainly not what we'd get using a Euclidean inner product, but we want the product to represent a cross product, not a Euclidean length squared.

Definition: A nonzero vector whose 'square' is zero is said to be **isotropic**.

³I apologize for choosing as my J the transpose of the usual symplectic matrix. I hope this doesn't cause too much confusion.

Definition: If the product of two nonzero vectors is zero, we shall refer to the product as an **isotrope**.

The Main Heuristic

Let X^α and Y^α be nonzero spinors forming the isotrope $Y_\alpha X^\alpha = 0$; then we know that there exists a nonzero scale factor κ , say, such that

$$Y_\alpha = \kappa X_\alpha. \quad (29)$$

Say we wish to solve for κ . One way to do this is to have a third spinor Z^α satisfying two equations

$$Y_\alpha Z^\alpha = a, \quad (30a)$$

$$X_\alpha Z^\alpha = b. \quad (30b)$$

Then we multiply (29) through by Z^α and sum, yielding the relation $a = \kappa b$ to solve for κ .

We shall see that in solving real problems, either an isotrope will arise naturally in a given problem, or we shall benefit by finding a way to construct one, no matter how artificial that construction may appear.

2 Appendix: Spinor Cheat Sheet

Given spinor $A^\alpha \mapsto \begin{bmatrix} A^1 \\ A^2 \end{bmatrix}$, then

$$A_\alpha \mapsto [A_1, A_2] = [-A^2, A^1]. \quad (31)$$

Given spinor $B_\alpha \mapsto [B_1, B_2]$, then

$$B^\alpha \mapsto \begin{bmatrix} B_2 \\ -B_1 \end{bmatrix}. \quad (32)$$

Given spinor $\hat{A}^\alpha \mapsto \begin{bmatrix} \bar{A}^1 \\ A^2 \end{bmatrix}$, then

$$\hat{A}_\alpha \mapsto [A_1, \bar{A}_2]. \quad (33)$$

References

- [1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer. (1999).