Proving Standard Triangle Theorems with Isotropic Spinors

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Abstract

Standard Triangle Theorems, such as the medians of a triangle are concurrent, are proved with Isotropic Spinors.

Introduction:
Euclidean geometry (the stuff we did in high school) has many powerful theorems and postulates that allow one to prove the theorems in this paper with little apparent effort. Using the spinor method – or what is really in this paper a vector method for the plane – takes a bit more effort. Vector methods are not as blessed with the accumulative effect of theorem upon theorem to form evermore powerful theorems to solve problems. Vector methods seem to always begin at the beginning. But they work anyway. The powerful theorems of Ceva and Menelaus are not ours to use here, but we make due.

Definition 1:
A **median** is a line segment connecting a vertex of a triangle to the midpoint of its opposite side.

Midpoint Lemma
The midpoint M^α of a line segment in terms of the endpoints A^α and B^α is

$$M^\alpha = \tfrac{1}{2}(A^\alpha + B^\alpha)\,. \tag{1}$$

Proof:
In words: First go to point A and then go halfway between A and B:

$$M^\alpha = A^\alpha + \tfrac{1}{2}(B^\alpha - A^\alpha) = \tfrac{1}{2}(A^\alpha + B^\alpha)\,. \tag{2}$$

Theorem 1
The medians of a triangle are concurrent (at the so-called *centroid*), and the centroid divides the medians in ratio of 2:1.

Proof of Theorem 1:
In the figure below, $\overline{AE}$ and $\overline{CD}$ are medians, meeting at point G. The point F is the extension of $\overline{BG}$ to $\overline{AC}$.

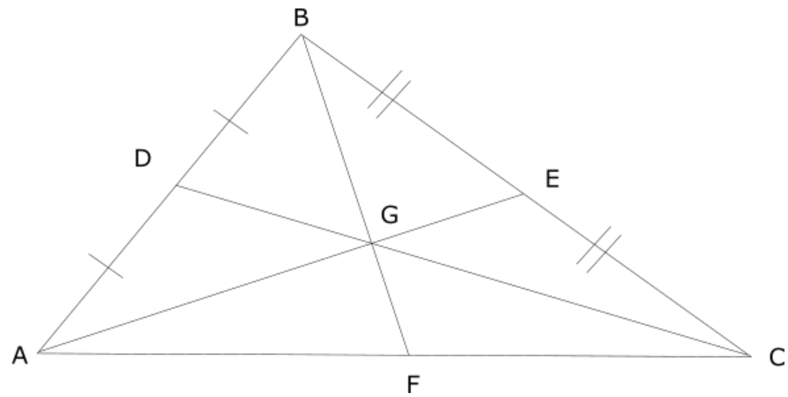

Figure 1. The Medians of a Triangle are Concurrent.

Our job now is to show that F is the midpoint of $\overline{AC}$, which would take the spinor form

$$F^\alpha = \tfrac{1}{2}(A^\alpha + C^\alpha)\,. \tag{3}$$

We now need to give algebraic expression to what we know about points D, E, F, and G, beginning with D and E:

$$D^\alpha = \tfrac{1}{2}(A^\alpha + B^\alpha)\,, \tag{4a}$$
$$E^\alpha = \tfrac{1}{2}(B^\alpha + C^\alpha)\,. \tag{4b}$$

Now we express that the point G belong to three line segments, and that F lies on $\overline{AC}$:

$$(C_\alpha - D_\alpha)(G^\alpha - C^\alpha) = 0\,, \tag{5a}$$
$$(A_\alpha - E_\alpha)(G^\alpha - A^\alpha) = 0\,, \tag{5b}$$
$$(B_\alpha - F_\alpha)(G^\alpha - B^\alpha) = 0\,, \tag{5c}$$
$$(A_\alpha - F_\alpha)(A^\alpha - C^\alpha) = 0\,. \tag{5d}$$

Before picking a strategy to solve this problem, I like to consider how to get more useful information out of the given information, particularly from (4a) and (4b). I can foresee a problem in mixing my constraints to work together because some of them have factors of $\frac{1}{2}$ and the rest don't. It's a clear case of the 'halfs and the half-nots'. So, let's add together (4a) and (4b) to get

$$E^\alpha + D^\alpha = B^\alpha + \tfrac{1}{2}(A^\alpha + C^\alpha)\,. \tag{6a}$$

Now that looks promising, because the expression $\frac{1}{2}(A^\alpha + C^\alpha)$ represents the midpoint on the line segment $\overline{AC}$, and that's where we want to show that the point F lies. Let's label this midpoint M^α for future reference:

$$M^\alpha = \tfrac{1}{2}(A^\alpha + C^\alpha)\,. \tag{6b}$$

And rewrite (6a) as

$$E^\alpha + D^\alpha = B^\alpha + M^\alpha\,. \tag{6c}$$

A very successful strategy I have used in situations like this in the past has been to eliminate interior points from the system of equations and then winnow the system down and see what happens. In this case, the point I want to try to get rid of is G. To that end, let's open up Equations (5a), (5b), and (5c):

$$(C_\alpha - D_\alpha)G^\alpha = (C_\alpha - D_\alpha)C^\alpha = -D_\alpha C^\alpha\,, \tag{7a}$$

$$(A_\alpha - E_\alpha)G^\alpha = (A_\alpha - E_\alpha)A^\alpha = -E_\alpha A^\alpha\,, \tag{7b}$$

$$(B_\alpha - F_\alpha)G^\alpha = (B_\alpha - F_\alpha)B^\alpha = -F_\alpha B^\alpha\,. \tag{7c}$$

So we ask, How do we combine constraint (6c) to these three equations? The answer: By adding (7a) and (7b) and (7c) together to form the sum:

$$[(C_\alpha + A_\alpha - F_\alpha) - (D_\alpha + E_\alpha - B_\alpha)]G^\alpha = -D_\alpha C^\alpha - E_\alpha A^\alpha - F_\alpha B^\alpha\,. \tag{8}$$

Using (6b) and (6c), the LHS of (8) becomes

$$[\,(2M_\alpha - F_\alpha) - M_\alpha]G^\alpha \mapsto (M_\alpha - F_\alpha)G^\alpha\,. \tag{9}$$

The problem now is how to simplify the RHS of (8)? Clearly there are too many points in it. We have to develop a strategy on how to remove points from this expression. Do we keep the midpoints D^α and E^α in favor of the vertices, or the other way around? For me, the term $F_\alpha B^\alpha$ is key to answering this, because it has a vertex point in it. Therefore, to proceed, we will keep the vertices in favor of the midpoints D^α and E^α.

Let's begin by rewriting the RHS of (8) as $D^\alpha C_\alpha + E^\alpha A_\alpha - F_\alpha B^\alpha$. Then from (4a) and (4b) we get

$$D^\alpha C_\alpha = \tfrac{1}{2}(A^\alpha + B^\alpha)C_\alpha\,, \tag{10a}$$

$$E^\alpha A_\alpha = \tfrac{1}{2}(B^\alpha + C^\alpha)A_\alpha\,. \tag{10b}$$

From which we get that

$$D^\alpha C_\alpha + E^\alpha A_\alpha = B^\alpha M_\alpha\,. \tag{11}$$

Thus Eq. (8) becomes

$$(M_\alpha - F_\alpha)G^\alpha = B^\alpha(M_\alpha - F_\alpha)\,. \tag{12}$$

Clearly, a better strategy than to algebraically eliminate G^α from the system is rather to use G^α to make the following critical geometrical argument. Simplifying (12), we get

$$(M_\alpha - F_\alpha)(G^\alpha - B^\alpha) = 0\,. \tag{13}$$

So, either the vectors $M^\alpha - F^\alpha$ and $G^\alpha - B^\alpha$ are nonzero scalar multiples of each other, or one of them is the zero vector. However, if M^α and F^α are distinct points, then $M^\alpha - F^\alpha$ is a vector on $\overline{AC}$, and there is no way it is a scalar multiple of $G^\alpha - B^\alpha$. The only option left is that one of $M^\alpha - F^\alpha$ or $G^\alpha - B^\alpha$ is the zero vector. But since $G^\alpha - B^\alpha \neq 0^\alpha$ then $M^\alpha - F^\alpha = 0^\alpha$ and we conclude that $F^\alpha = M^\alpha$.

Our next task is to prove that the centroid G^α divides the medians into ratios of 2:1. For this proof, any one of the three medians will do. So let's choose $\overline{CGD}$. Consider the following relation:

$$(G^\alpha - C^\alpha) = \lambda(D^\alpha - G^\alpha)\,. \tag{14}$$

If we can show that λ is either 2 or 1/2, we'd have our proof. To solve this for λ we need another equation for G^α. G^α is the intersection of two line segments, and I have a standard way to solve for that point as functions of the four end points. See Appendix 1 for this formula and derivation. So,

$$G^\alpha = D^\alpha + \mu(C^\alpha - D^\alpha)\,, \tag{15}$$

where

$$\mu = \frac{(A^\alpha - D^\alpha)(E_\alpha - A_\alpha)}{(C^\alpha - D^\alpha)(E_\alpha - A_\alpha)}\,. \tag{16}$$

Eliminating D^α and E_α from this, and simplifying, we get[1]

$$\mu = \frac{A^\alpha C_\alpha + C^\alpha B_\alpha + B^\alpha A_\alpha}{3(A^\alpha C_\alpha + C^\alpha B_\alpha + B^\alpha A_\alpha)} = \frac{1}{3}\,, \tag{17}$$

an interesting number, this μ.[2] Rewriting (14) as

$$G^\alpha(1+\lambda) = C^\alpha + \lambda D^\alpha\,, \tag{18}$$

and substituting G^α from (15), we get

$$[D^\alpha + \mu(C^\alpha - D^\alpha)](1+\lambda) = C^\alpha + \lambda D^\alpha\,. \tag{19}$$

Now, multiply this equation either by C_α or by D_α and simplify to get

$$\mu(1+\lambda) = 1\,. \tag{20}$$

Therefore, using (17) to solve for λ, we get $\lambda = 2$, and that proves the claim.

Theorem 2
The perpendicular bisectors of a triangle are concurrent (at the so-called *circumcenter*).

[1] Note: Ignoring sign, the number $A^\alpha C_\alpha + C^\alpha B_\alpha + B^\alpha A_\alpha$ is twice the area of the $\triangle ABC$.
[2] Hindsight is perfect, isn't it? We could have used this μ instead of λ to establish the result, but starting with Eq. (14) seems more natural. Anyway: 1/3 : 2/3 :: 1 : 2.

Proof:
In the figure below, D is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{BC}$. The perpendicular bisectors from those midpoints meet at point G. A line segment is drawn from G to the midpoint F of $\overline{AC}$. Show that $\overline{GF}$ is perpendicular to $\overline{AC}$.

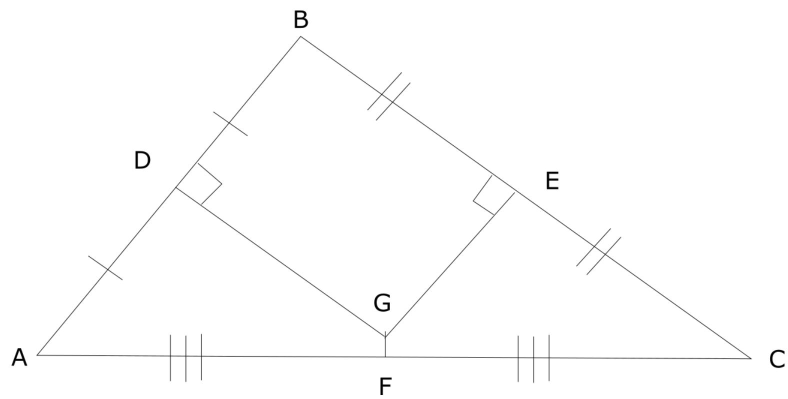

Figure 2. The perpendicular bisectors of a triangle are concurrent.

This is my chance to show that isotropic spinors can also work well with perpendicular lines/vectors, almost as easily as it does with parallel lines/vectors. Before we can get into the details of this proof, we must see how to deal with perpendicular objects in the formalism.

The idea is simple: Take a pair of mutually perpendicular lines, and rotate one of them by 90° counterclockwise (though the direction of rotation won't matter). Then the new pair of lines are parallel. The same thing works for vectors. The way to rotate vector A^α by 90° counterclockwise is to apply the operator

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{21}$$

on the left, which is appropriate for matrix multiplication on a column vector.[3]

$$JA^\alpha = \begin{pmatrix} -A^2 \\ A^1 \end{pmatrix} \tag{22}$$

Fortunately, we won't need to look at components in this proof. However, we will have to look at them to establish the following identity, which we will need.

Lemma 1
The product through J is symmetric. Let X^α and Y^α be any two vectors, then

$$X_\alpha J Y^\alpha = (-X^2, X^1) \begin{pmatrix} -Y^2 \\ Y^1 \end{pmatrix} = X^2 Y^2 + X^1 Y^1\,. \tag{23}$$

[3]Note: Of course J can operate on the right, but only on a compatible matrix, such as a 2×2 or a 1×2 matrix (that is, a row matrix).

But

$$Y_\alpha J X^\alpha = (-Y^2, Y^1)\begin{pmatrix}-X^2\\ X^1\end{pmatrix} = Y^2X^2 + Y^1X^1\,. \tag{24}$$

Therefore,

$$X_\alpha J Y^\alpha = Y_\alpha J X^\alpha\,. \tag{25}$$

As a corollary, the square length of a vector $\mathbf{X}$ is given by

$$\mathbf{X}^2 = X_\alpha J X^\alpha\,. \tag{26}$$

Definition 2:
A *virtual emplacement* is an identity operation on an expression that emplaces a quantity and then subtracts it out or divides it out, whichever case is needed to leave the expression's value unchanged. In other words, it's like adding in zero or multiplying by unity. Examples:

$$X_\alpha Y^\alpha = \tfrac{1}{2}X_\alpha(2Y^\alpha)\,, \tag{27}$$

and

$$X_\alpha Y^\alpha = X_\alpha(Y^\alpha - Z^\alpha) + X_\alpha(Z^\alpha)\,, \tag{28}$$

Here's an important virtual emplacement:

$$X_\alpha Y^\alpha = X_\alpha(Y^\alpha + X^\alpha)\,. \tag{29}$$

(Hint: $X_\alpha X^\alpha = 0$.)

As a test of your ability to do orthogonality in sympletic spinors, you could prove the Pythagoream Theorem at this point using this J notation. I give the proof in Appendix (2).

Now to give algebraic representation for all the constraints given. We begin with the three midpoint constraints:

$$D^\alpha = \tfrac{1}{2}(A^\alpha + B^\alpha)\,, \tag{30a}$$
$$E^\alpha = \tfrac{1}{2}(B^\alpha + C^\alpha)\,, \tag{30b}$$
$$F^\alpha = \tfrac{1}{2}(A^\alpha + C^\alpha)\,. \tag{30c}$$

Taking a hint from the previous problem, let's add these three equations together, to get

$$D^\alpha + E^\alpha + F^\alpha = A^\alpha + B^\alpha + C^\alpha\,. \tag{31}$$

We also have two constraints of perpendicularity to declare:

$$(A_\alpha - B_\alpha)J(G^\alpha - D^\alpha) = 0\,, \tag{32a}$$
$$(B_\alpha - C_\alpha)J(G^\alpha - E^\alpha) = 0\,. \tag{32b}$$

What we are to show is that

$$(A_\alpha - C_\alpha)J(G^\alpha - F^\alpha) = 0\,, \tag{33}$$

or what is equivalent, letting

$$\Phi = (A_\alpha - C_\alpha)JG^\alpha\,, \tag{34}$$

show that

$$\Phi = (A_\alpha - C_\alpha)JF^\alpha\,. \tag{35}$$

Let's begin by opening up (32a) and (32b):

$$(A_\alpha - B_\alpha)JG^\alpha = (A_\alpha - B_\alpha)JD^\alpha\,, \tag{36a}$$
$$(B_\alpha - C_\alpha)JG^\alpha = (B_\alpha - C_\alpha)JE^\alpha\,. \tag{36b}$$

If we add Equations (36a) and (36b) and use that result with (34), we get

$$\Phi = (A_\alpha - B_\alpha)JD^\alpha + (B_\alpha - C_\alpha)JE^\alpha\,. \tag{37}$$

Multiplying through by 2 and using (30a) and (30b), we get

$$2\Phi = (A_\alpha - B_\alpha)J(A^\alpha + B^\alpha) + (B_\alpha - C_\alpha)J(B^\alpha + C^\alpha)\,. \tag{38}$$

After expanding the products in the RHS of (38), and simplifying, we get

$$2\Phi = A_\alpha JA^\alpha - C_\alpha JC^\alpha\,, \tag{39}$$

Now we virtually emplace whatever we need to arrive at a form appropriate to (35):

$$\begin{aligned}2\Phi &= A_\alpha J(A^\alpha + C^\alpha - C^\alpha) - C_\alpha J(A^\alpha + C^\alpha - A^\alpha) \\ &= A_\alpha J(A^\alpha + C^\alpha) - \cancel{A_\alpha JC^\alpha} - C_\alpha J(A^\alpha + C^\alpha) + \cancel{C_\alpha JA^\alpha} \\ &= A_\alpha J(A^\alpha + C^\alpha) - C_\alpha J(A^\alpha + C^\alpha) \\ &= (A_\alpha - C_\alpha)J(A^\alpha + C^\alpha)\,. \end{aligned} \tag{40}$$

Multiplying through by $\frac{1}{2}$ and using (30c):

$$\Phi = (A_\alpha - C_\alpha)J[\tfrac{1}{2}(A^\alpha + C^\alpha)] = (A_\alpha - C_\alpha)JF^\alpha\,, \tag{41}$$

which is what we were to show. It turns out that this version of the proof did not require the elegant Equation (31), suggesting to me that there may be a more elegant proof than presented here.

To follow-up on the heuristics used here: We were tasked with starting with one expression and needing to end up with another expression. Along the way we were required to incorporate the given information (what I called the 'constraints') and use a virtual emplacement here and there. In the heuristic sense, we can think of virutal emplacement operations and cancelling operations as roughly inverse to each other. This is evident when you run the steps of the proof backward for comparison.

Theorem 3
The altitudes of a triangle are concurrent (at the so-called *orthocenter*).

Proof:
In the figure below, $\overline{DC}$ is perdendicular to $\overline{AB}$ and $\overline{AE}$ is perdendicular $\overline{BC}$. $\overline{DC}$ and $\overline{AE}$ are called *altitudes* of the triangle, and they meet at point G. Line segment $\overline{BG}$ is extended to $\overline{AC}$, meeting it at point F. Show that $\overline{BF}$ is perpendicular to $\overline{AC}$.

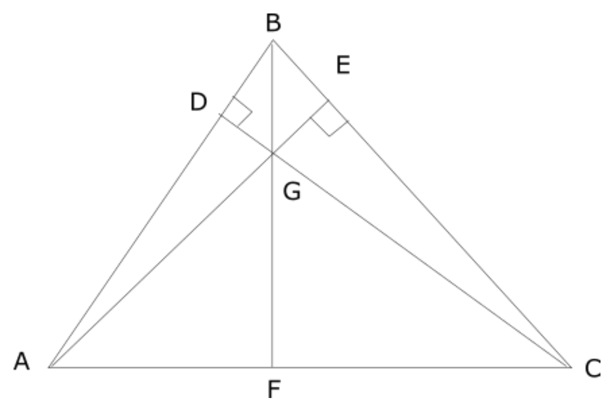

Figure 3. The altitudes of a triangle are concurrent.

Now, since we are not required to prove anything specific about the points D, E, and F, we will avoid using them in the algebraic constraints. So, we now declare all our constraints, starting with, say, the orthogonality relations:

$$(A_\alpha - B_\alpha)J(G^\alpha - C^\alpha) = 0\,, \tag{42a}$$
$$(B_\alpha - C_\alpha)J(G^\alpha - A^\alpha) = 0\,. \tag{42b}$$

What we are to show is that $\overline{BG}$ is perpendicular to $\overline{AC}$,

$$(A_\alpha - C_\alpha)J(B^\alpha - G^\alpha) = 0\,, \tag{43}$$

or what is equivalent to it:

$$(A_\alpha - C_\alpha)JB^\alpha = (A_\alpha - C_\alpha)JG^\alpha\,. \tag{44}$$

Before committing to a strategy, let's tinker with the constraints to see what we get. Let's first open up (42a) and (42b):

$$(A_\alpha - B_\alpha)JG^\alpha = (A_\alpha - B_\alpha)JC^\alpha\,, \tag{45a}$$
$$(B_\alpha - C_\alpha)JG^\alpha = (B_\alpha - C_\alpha)JA^\alpha\,. \tag{45b}$$

Adding (45a) and (45b), we get:

$$(A_\alpha - C_\alpha)JG^\alpha = (A_\alpha - B_\alpha)JC^\alpha + (B_\alpha - C_\alpha)JA^\alpha \,. \tag{46}$$

If we can turn the RHS of (46) into the LHS of (44), we are finished.

$$\begin{aligned}(A_\alpha - B_\alpha)JC^\alpha + (B_\alpha - C_\alpha)JA^\alpha &= \cancel{A_\alpha JC^\alpha} - B_\alpha JC^\alpha + B_\alpha JA^\alpha - \cancel{C_\alpha JA^\alpha} \\ &= B_\alpha J(A^\alpha - C^\alpha) \\ &= (A_\alpha - C_\alpha)JB^\alpha \,. \end{aligned} \tag{47}$$

And that concludes the proof.

An important heuristic point should be made here. This problem proves that not every point in the diagram requires inclusion in the set of constraints needed to solve the problem. In fact, including unneeded information could be detrimental, if not fatal, to finding a correct solution.

Definition 3
We can think of the spinors A^α, B^α, C^α, D^α, etc as *spinor flavors* and they may *oscillate* amongst themselves under certain circumstances, which we show now. Consider the expression

$$(D_\alpha - A_\alpha)(D^\alpha - C^\alpha) \mapsto (D_\alpha - A_\alpha)(A^\alpha - C^\alpha) \,. \tag{48}$$

The D^α has oscillated to an A^α, and the two expressions are actually equal. Proof:

$$\begin{aligned}(D_\alpha - A_\alpha)(D^\alpha - C^\alpha) &= (D_\alpha - A_\alpha)[(D^\alpha - A^\alpha) + (A^\alpha - C^\alpha)] \\ &= (D_\alpha - A_\alpha)(D^\alpha - A^\alpha) + (D_\alpha - A_\alpha)(A^\alpha - C^\alpha) \\ &= (D_\alpha - A_\alpha)(A^\alpha - C^\alpha) \,. \end{aligned} \tag{49}$$

I introduce this term because, by use of it, I can save a lot of tedious and uninteresting writing in proofs. Also, the notion lends itself heuristically. By the way, lower index spinors can be oscillated too.

Convention
The length of line segment $\overline{AB}$ (or the distance from point A to point B) is represented by AB. I state without proof that the length of $\overline{BA}$ is the same as the length of $\overline{AB}$, thus, $AB = BA$.

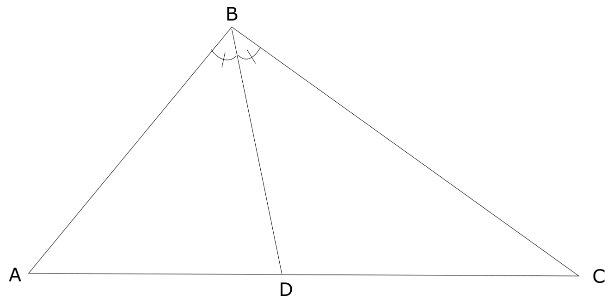

Figure 4. The line segment $\overline{BD}$ bisects the vertex angle B.

Theorem 4: The Angle Bisector Theorem
The bisector of a vertex angle divides the opposite side in the same ratio as the ratio of the sides of the chosen vertex angle. (Refer to Figure. 4.) In other words,

$$\frac{AD}{BA} = \frac{CD}{BC}\,. \tag{50}$$

Proof:
I got the starting idea for this proof from *Treatise on Plane Geometry Through Geometric Algebra*, [1]. Since $\angle CBD = \angle ABD$, the point D^α can be reached by starting at point B^α and traveling equal amounts along the unit vectors of the vectors $A^\alpha - B^\alpha$ and $C^\alpha - B^\alpha$. We can represent this fact in symbols by

$$D^\alpha = B^\alpha + \lambda\left[\frac{A^\alpha - B^\alpha}{|A^\alpha - B^\alpha|} + \frac{C^\alpha - B^\alpha}{|C^\alpha - B^\alpha|}\right]. \tag{51}$$

The problem of this λ is one of two choices: Either we can solve for it, or divide it out. My initial thought at dealing with λ was to subtract B^α from both sides and multiply through by first $A^\alpha - B^\alpha$ and get one equation in λ and then by $C^\alpha - B^\alpha$ to get another equation in λ and then take their ratios to cancel out λ. But this had the effect of strongly coupling points D and B, but to no useful purpose, because the number BD does not even appear in (50). Before continuing the heuristics, let's make a simplification

$$|A^\alpha - B^\alpha| = a \quad \text{and} \quad |C^\alpha - B^\alpha| = b\,. \tag{52}$$

Now (51) becomes

$$D^\alpha = B^\alpha + \lambda\left[\frac{A^\alpha - B^\alpha}{a} + \frac{C^\alpha - B^\alpha}{b}\right]. \tag{53}$$

So, instead of coupling D and B, let's couple D and A by subtracting A^α from both sides of (53) and then multiplying through by $A_\alpha - C_\alpha$. Since $(A_\alpha -$

$C_\alpha)(D^\alpha - A^\alpha) = 0$, this gives us an equation to solve for λ.

$$0 = (A_\alpha - C_\alpha)(B^\alpha - A^\alpha) + \lambda \left[\frac{(A_\alpha - C_\alpha)(A^\alpha - B^\alpha)}{a} + \frac{(A_\alpha - C_\alpha)(C^\alpha - B^\alpha)}{b} \right] . \tag{54}$$

On factoring the RHS, we get

$$0 = (A_\alpha - C_\alpha)(B^\alpha - A^\alpha)\big[1 + \lambda\big(\frac{-1}{a} + \frac{-1}{b}\big)\big] , \tag{55}$$

where we oscillated the C^α in (54) to A^α prior to factoring. (Please note that I refer to the C only with the upper index of α). From which we get that

$$1 + \lambda\big(\frac{-1}{a} + \frac{-1}{b}\big) = 0 ,$$

so that

$$\lambda = \frac{ab}{a+b} . \tag{56}$$

Using this result in (53), we get[4]

$$D^\alpha = \frac{1}{a+b}[bA^\alpha + aC^\alpha] . \tag{57}$$

Now, subtracting A^α from both sides and taking the vector norm of the results, we get

$$|D^\alpha - A^\alpha| = \frac{a}{a+b}|C^\alpha - A^\alpha| . \tag{58}$$

Now we make the substitutions $|D^\alpha - A^\alpha| \mapsto AD$, $|C^\alpha - A^\alpha| \mapsto CA$, $a \mapsto BA$, $b \mapsto BC$, to get, after a bit of algebra,

$$\frac{AD}{BA} = \frac{CD}{BC} , \tag{59}$$

which is what we were to show.

Theorem 5
The vertex bisectors of a triangle meet in a point, called the *incenter*. In the figure below, $\overline{BF}$ is a bisector of the angle at vertex B, and $\overline{CD}$ is a bisector of the angle at vertex C. These two line segments meet at point G. $\overline{AG}$ is extended until it hits $\overline{BC}$ at E. Our task is to show that the segment $\overline{AE}$ bisects the angle at vertex A.

[4] D^α appears as a 'weighted average' of the endpoints of the line segment it resides on.

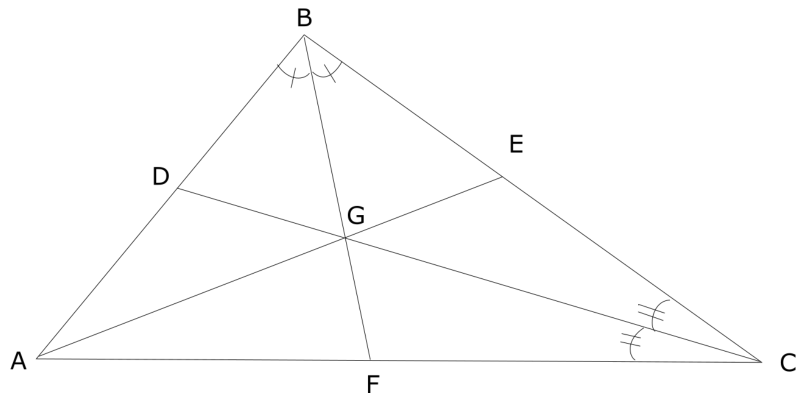

Figure 5. The vertex bisectors of a triangle meet in a point, called the *incenter*.

Proof:
We're going to form unit vectors in this proof, like we did in the last proof. Also, we'll make the following substitutions wherever appropriate: $a = |A^\alpha - C^\alpha|$, $b = |B^\alpha - C^\alpha|$, $c = |A^\alpha - B^\alpha|$. Thus, we can write the vector from C^α to G^α (that is, $G^\alpha - C^\alpha$) as:

$$G^\alpha - C^\alpha = \lambda \left[\frac{A^\alpha - C^\alpha}{a} + \frac{B^\alpha - C^\alpha}{b} \right] , \tag{60}$$

and for $G^\alpha - B^\alpha$

$$G^\alpha - B^\alpha = \mu \left[\frac{A^\alpha - B^\alpha}{c} + \frac{C^\alpha - B^\alpha}{b} \right] . \tag{61}$$

What we need to show is that for some τ,

$$G^\alpha - A^\alpha = \tau \left[\frac{B^\alpha - A^\alpha}{c} + \frac{C^\alpha - A^\alpha}{a} \right] . \tag{62}$$

Let's try the method of undetermined coefficients. Let

$$G^\alpha - A^\alpha = \sigma(B^\alpha - A^\alpha) + \gamma(C^\alpha - A^\alpha) . \tag{63}$$

and solve for σ and γ. We can start this process by first multiplying (63) through by $(C^\alpha - A^\alpha)$ and then by $(B^\alpha - A^\alpha)$, to get

$$\sigma = \frac{(C_\alpha - A_\alpha)(G^\alpha - A^\alpha)}{(C_\alpha - A_\alpha)(B^\alpha - A^\alpha)} \tag{64a}$$

$$\gamma = \frac{(B_\alpha - A_\alpha)(G^\alpha - A^\alpha)}{(B_\alpha - A_\alpha)(C^\alpha - A^\alpha)} . \tag{64b}$$

Now, in the numerator of the σ equation, A^α oscillates to C^α, and in the numerator of the γ equation, A^α oscillates to B^α.

$$\sigma = \frac{(C_\alpha - A_\alpha)(G^\alpha - C^\alpha)}{(C_\alpha - A_\alpha)(B^\alpha - A^\alpha)} \tag{65a}$$

$$\gamma = \frac{(B_\alpha - A_\alpha)(G^\alpha - B^\alpha)}{(B_\alpha - A_\alpha)(C^\alpha - A^\alpha)}\,. \tag{65b}$$

Doing this allows us to use the given information in Equations (60) and (61) to further analyze σ and γ, and eliminating G^α as we go. So, multiplying (60) through by $(C_\alpha - A_\alpha)$ and (61) through by $(B_\alpha - A_\alpha)$ and employing those results, we can now write:

$$\sigma = \frac{\lambda}{b}\,\frac{(C_\alpha - A_\alpha)(B^\alpha - C^\alpha)}{(C_\alpha - A_\alpha)(B^\alpha - A^\alpha)} \tag{66a}$$

$$\begin{aligned}\gamma &= \frac{\mu}{b}\,\frac{(B_\alpha - A_\alpha)(C^\alpha - B^\alpha)}{(B_\alpha - A_\alpha)(C^\alpha - A^\alpha)}\\ &= \frac{\mu}{b}\,\frac{(C_\alpha - A_\alpha)(C^\alpha - B^\alpha)}{(B_\alpha - A_\alpha)(C^\alpha - A^\alpha)}\,,\end{aligned} \tag{66b}$$

where we oscillated the $B_\alpha \mapsto C_\alpha$ in the numerator of γ equation to facilitate comparison of these fractions. Dividing σ by γ, we get

$$\frac{\sigma}{\gamma} = \frac{\lambda/b}{\mu/b} = \frac{\lambda}{\mu}\,. \tag{67}$$

It seems we need to find this ratio of λ to μ. So, multiply (60) by $(B_\alpha - C_\alpha)$ to get

$$(B_\alpha - C_\alpha)(G^\alpha - C^\alpha) = \frac{\lambda}{a}(B_\alpha - C_\alpha)(A^\alpha - C^\alpha)\,. \tag{68}$$

and (61) by $(C_\alpha - B_\alpha)$ to get

$$(C_\alpha - B_\alpha)(G^\alpha - B^\alpha) = \frac{\mu}{c}(C_\alpha - B_\alpha)(A^\alpha - B^\alpha)\,. \tag{69}$$

And oscillating B^α to C^α in this last equation, we get

$$(C_\alpha - B_\alpha)(G^\alpha - C^\alpha) = \frac{\mu}{c}(C_\alpha - B_\alpha)(A^\alpha - C^\alpha)\,. \tag{70}$$

Now, comparing (68) and (70), we get

$$\frac{\lambda}{a} = \frac{\mu}{c}\,. \tag{71}$$

Combining (64b) and (71), we get

$$\frac{\sigma}{\gamma} = \frac{a}{c}\,. \tag{72}$$

Using this in (63), we get

$$\begin{aligned} G^\alpha - A^\alpha &= \gamma \frac{a}{c}(B^\alpha - A^\alpha) + \gamma(C^\alpha - A^\alpha) \\ &= \gamma a \Big[\frac{B^\alpha - A^\alpha}{c} + \frac{C^\alpha - A^\alpha}{a}\Big] \\ &= \gamma a \Big[\frac{B^\alpha - A^\alpha}{|B^\alpha - A^\alpha|} + \frac{C^\alpha - A^\alpha}{|C^\alpha - A^\alpha|}\Big]. \end{aligned} \tag{73}$$

So, finally, we found our scale factor τ: $\tau = \gamma a$.

Problem 1
Find the value of the λ parameter in (60) in terms of the values of a, b, c.

We begin by eliminating G^α between (60) and (61). So, we subtract (61) from (60), yielding,

$$B^\alpha - C^\alpha = \lambda \left[\frac{A^\alpha - C^\alpha}{a} + \frac{B^\alpha - C^\alpha}{b}\right] - \mu \left[\frac{A^\alpha - B^\alpha}{c} + \frac{C^\alpha - B^\alpha}{b}\right]. \tag{74}$$

Next, we multiply this through by $A_\alpha - B_\alpha$, and afterward oscillate A^α to C^α, then cancel out $(A_\alpha - B_\alpha)(B^\alpha - C^\alpha)$, to get

$$1 = \lambda \left[\frac{1}{a} + \frac{1}{b}\right] + \frac{\mu}{b}. \tag{75}$$

Now, using (71) we can obtain one equation in the variable λ, which can be solved to get

$$\lambda = \frac{ab}{a+b+c}. \tag{76}$$

Theorem 6
An angle inscribed in a semicircle is right.

Proof:
In the figure below, $\angle B$ is an angle inscribed in a semicircle. $\triangle ABC$ has $\overline{AC}$ is a diameter of the circle. We are to show that angle at B is right. We've place the origin of the circle at O. Obviously, $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are all radii of the circle, and thus they all have the same length — and square length as well. The vector $C^\alpha - O^\alpha = -(A^\alpha - O^\alpha)$. But since we will take O^α as the zero vector, we can write

$$C^\alpha = -A^\alpha. \tag{77}$$

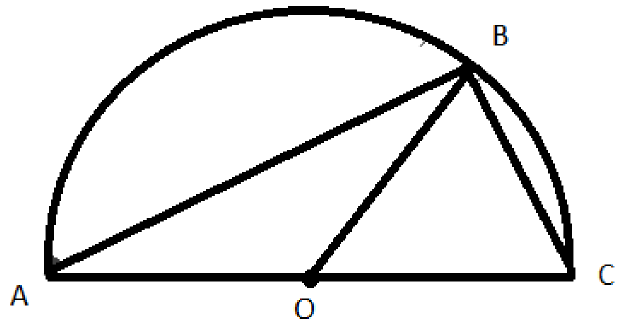

Figure 6. An angle inscribed in a semicircle.

We are to show that the angle at B is right, which requires us to show that

$$(A_\alpha - B_\alpha)J(B^\alpha - C^\alpha) = 0\,. \tag{78}$$

Using (77) in (78), we get:

$$\begin{aligned}(A_\alpha - B_\alpha)J(B^\alpha - C^\alpha) &= (A_\alpha - B_\alpha)J(B^\alpha + A^\alpha)\\ &= \cancel{A_\alpha J B^\alpha} + \cancel{A_\alpha J A^\alpha} - \cancel{B_\alpha J B^\alpha} - \cancel{B_\alpha J A^\alpha}\\ &= 0\,. \end{aligned} \tag{79}$$

And we are finished.

Conclusion
The isotropic nature of the spinors we defined has afforded us a convenient means to deal with questions of collinearity of points and the perpendicularity of lines. The proofs contained here is well within the reach of the high school student who is already familiar with vector algebra. Only a little preparation to get the student familiar with the symplectic inner product should be enough foundation.

As a final thought, one practical advantage of using vector methods in preference to synthetic methods of doing triangle problems in plane geometry is that the locations of special points, such as incenters, orthocenters, centroids, and the like, are easily calculated once the vertices are given planar coordinates.

1 Appendix: Solving for a Point of Intersection

Refer to Figure 1.

$$G^\alpha = D^\alpha + \mu(C^\alpha - D^\alpha)\,, \tag{80a}$$

$$G^\alpha = A^\alpha + \tau(E^\alpha - A^\alpha)\,, \tag{80b}$$

Setting these two equations equal and simplifying, we get

$$\mu(C^\alpha - D^\alpha) = (A^\alpha - D^\alpha) + \tau(E^\alpha - A^\alpha) \tag{81}$$

Multiplying this through by $(E_\alpha - A_\alpha)$ and solving for μ, we get

$$\mu = \frac{(A^\alpha - D^\alpha)(E_\alpha - A_\alpha)}{(C^\alpha - D^\alpha)(E_\alpha - A_\alpha)}\,. \tag{82}$$

2 Appendix: The Pythagorean Theorem

Theorem The sum of the squares of the sides of a right triangle equals the square of the hypotenuse.

Proof:
In the figure below, side $\overline{AC}$ is perpendicular to side $\overline{CB}$. $\overline{AB}$ is called the *hypotenuse* of the right triangle. Prove that

$$AB^2 = AC^2 + CB^2\,. \tag{83}$$

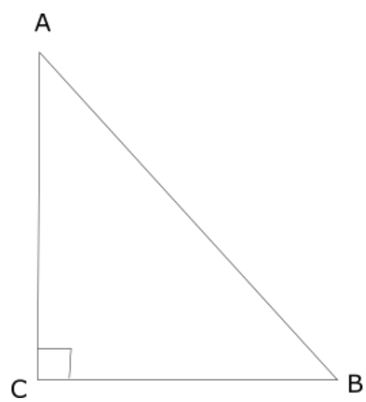

Figure: A right triangle.

In our spinor formalism we need to show that

$$(A_\alpha - B_\alpha)J(A^\alpha - B^\alpha) = (A_\alpha - C_\alpha)J(A^\alpha - C^\alpha) + (C_\alpha - B_\alpha)J(C^\alpha - B^\alpha)\,. \tag{84}$$

We are given one orthogonality relation:

$$(A_\alpha - C_\alpha)J(C^\alpha - B^\alpha) = 0\,. \tag{85}$$

So we begin with a predictable virtual emplacement on both sides of the J of the expression $(A_\alpha - B_\alpha)J(A^\alpha - B^\alpha)$:

$$
\begin{aligned}
AB^2 &= (A_\alpha - B_\alpha)J(A^\alpha - B^\alpha) \\
&= [(A_\alpha - C_\alpha) + (C_\alpha - B_\alpha)J[(A^\alpha - C^\alpha) + (C^\alpha - B^\alpha)] \\
&= (A_\alpha - C_\alpha)J(A^\alpha - C^\alpha) + \cancelto{0}{(A_\alpha - C_\alpha)J(C^\alpha - B^\alpha)} \\
&\qquad + \cancelto{0}{(C_\alpha - B_\alpha)J(A^\alpha - C^\alpha)} + (C_\alpha - B_\alpha)J(C^\alpha - B^\alpha) \\
&= (A_\alpha - C_\alpha)J(A^\alpha - C^\alpha) + (C_\alpha - B_\alpha)J(C^\alpha - B^\alpha) \\
&= AC^2 + CB^2 \,. \qquad (86)
\end{aligned}
$$

And that completes the proof.

3 Appendix: Brief Summary of Isotropic Spinors

We start with any two vectors **A** and **B** in 2-space and represent them in matrix form as $\begin{pmatrix} A^1 \\ A^2 \end{pmatrix}$ and $\begin{pmatrix} B^1 \\ B^2 \end{pmatrix}$, respectively. These vectors also have a convenient component, or indicial, form as A^α and B^α, respectively, where α takes on values 1,2.

Now, you're probably familiar with the Euclidean inner product of two vectors, but we aren't going to use that inner product. (If we're going to capture the information in a cross product as a 'scalar product', this makes sense to use some other notion of inner product.) But for comparison's sake, we'll show what the Euclidean inner product of two vectors B^α, A^α looks like:

$$B_\alpha A^\alpha \equiv \sum_\alpha B_\alpha A^\alpha = B_1 A^1 + B_2 A^2 \,. \qquad (87)$$

In other words, to scalar-multiply two column vectors together, we will have to convert one of them to a row vector by this procedure: $B_\alpha = [JB^\alpha]^t = (B^\alpha)^t J^t$, where J is a 2×2 matrix used to lower the index of a vector. For the Euclidean case, J is the 2×2 identity matrix, and

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (B^1, B^2) \,, \qquad (88)$$

with the resulting familiar Euclidean square

$$B_\alpha B^\alpha = (B^1, B^2) \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} = (B^1)^2 + (B^2)^2 \,. \qquad (89)$$

However, for our symplectic inner product, we take the symplectic matrix $J =$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and get[5]

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (-B^2, B^1)\,. \tag{90}$$

where $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J^t$ is the transpose of the symplectic matrix.

Now we're ready to take the sympectic 'inner' product of vectors B^α and A^α:

$$B_\alpha A^\alpha = (B_1, B_2) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = (-B^2, B^1) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = -B^2 A^1 + B^1 A^2 \tag{91}$$

And, on taking this inner product in the opposite order, we get

$$A_\alpha B^\alpha = -A^2 B^1 + A^1 B^2 = -(-B^2 A^1 + B^1 A^2) \tag{92}$$

Comparing (91) to (92), we see that they are negatives of each other

$$A_\alpha B^\alpha = -B_\alpha A^\alpha \tag{93}$$

So our symplectic inner product acts like the antisymmetric cross product of two vectors.

Definition: So far the components of our two-component vectors have been real numbers or real-valued functions. But now we let them be complex-valued, in which case we refer to them as **spinors**. Spinors were introduced to physics by Paul Ehrenfest and Wolfgang Pauli in the 1920s to deal with the then novel notion of electron spin. We will not need that particular interpretation of them. Our use of them here is quite formal, but convenient, as we shall see.

Important Lemma
Let A^α and B^α be two spinors. Then their symplectic 'inner' product is

$$A_\alpha B^\alpha = \det \begin{bmatrix} A^1 & A^2 \\ B^1 & B^2 \end{bmatrix} = A^1 B^2 - B^1 A^2\,. \tag{94}$$

But what happens if we use the same vector in this scalar product? Let's try it. $A_\alpha A^\alpha = -A^2 A^1 + A^1 A^2 = 0$. And this is true whatever the components of A^α are! This is certainly not what we'd get using a Euclidean inner product, but we want the product to represent a cross product, not a Euclidean length squared.

Definition: A nonzero vector whose 'square' is zero is said to be **isotropic**.

[5] I apologize for choosing as my J the transpose of the usual sympletic matrix. I hope this doesn't cause too much confusion.

Definition: If the product of two nonzero vectors is zero, we shall refer to the product as an **isotrope**.

The Main Heuristic
Let X^α and Y^α be nonzero spinors forming the isotrope $Y_\alpha X^\alpha = 0$; then we know that there exists a nonzero scale factor κ, say, such that

$$Y_\alpha = \kappa X_\alpha \,. \tag{95}$$

Say we wish to solve for κ. One way to do this is to have a third spinor Z^α satisfiying two equations

$$Y_\alpha Z^\alpha = a \,, \tag{96a}$$

$$X_\alpha Z^\alpha = b \,. \tag{96b}$$

Then we multiply (95) through by Z^α and sum, yielding the relation $a = \kappa b$ to solve for κ.

We shall see that in solving real problems, either an isotrope will arise naturally in a given problem, or we shall benefit by finding a way to construct one, no matter how artificial that construction may appear.

4 Appendix: Spinor Cheat Sheet

Given spinor $A^\alpha \mapsto \begin{bmatrix} A^1 \\ A^2 \end{bmatrix}$, then

$$A_\alpha \mapsto [A_1, A_2] = [-A^2, A^1] \,. \tag{97}$$

Given spinor $B_\alpha \mapsto [B_1, B_2]$, then

$$B^\alpha \mapsto \begin{bmatrix} B_2 \\ -B_1 \end{bmatrix} \,. \tag{98}$$

Given spinor $\widehat{A}^\alpha \mapsto \begin{bmatrix} \bar{A}^1 \\ A^2 \end{bmatrix}$, then

$$\widehat{A}_\alpha \mapsto [A_1, \bar{A}_2] \,. \tag{99}$$

References

[1] R. G. Calvet, *Treatise on Plane Geometry Through Geometric Algebra*, (2007).