The Lambert W Function

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April 26, 2026

I feel that a visual representation of the Dirac algebra is of great
benefit, because it can provide an additional insight that
is not easily expressed with words or equations.
— David M. Goodmanson
['A graphical representation of the Dirac algebra',
American J. Phys., Vol. 64, No. 7,
July 1996, p. 870.]

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1 Introduction to the Lambert W Function

We'll start off with an overview of the Lambert W function. It was invented to unravel this product in variable x:

$$xe^x = A\,, \tag{1}$$

to solve for x. What a cool function, indeed! Hence,

$$x = W(xe^x) = W(A)\,, \tag{2}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with. And that's it, at least for us here.

The Lambert W function is to the expression xe^x what the natural logarithm is to e^x: They are both designed to extract from each the value/s of x.

I intend to present a brief introduction to the Lambert W function. Wikipedia (among other sources) do a better job. My point of view on them is that of an olympiad-style math problem-solver, which is rather specific — at least at this time. So, let's get started.

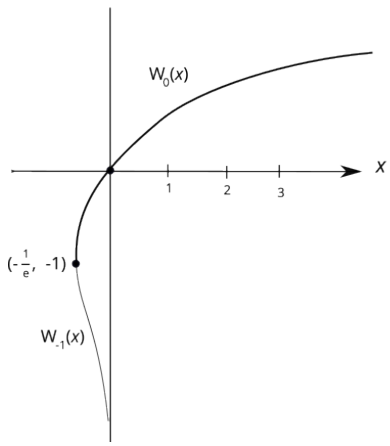

Figure 1. A rough sketch of the Lambert W function when x is real-valued. Emphasis is made of its multi-valued behavior near the origin. On the interval $-1/e < x < 0$, $W(x)$ has two distinct real values, depicted as $W_0(x)$ and $W_{-1}(x)$, which can be determined (approximately) with a computer program, such as WolframAlpha (using ProductLog[]).

From Wikipedia: Some special cases for the Lambert W function:

$W_0(e) = W(1 \cdot e^1) = 1.$

$W_0(-e^{-1}) = -1.$

$W_0(e^{1+e}) = e.$

$W_0(\frac{e^{1/2}}{2}) = 1/2.$

$W_0(\frac{e^{1/n}}{n}) = 1/n.$

$W_0(1) \equiv \Omega = e^{-W_0(1)} = -\ln W_0(1) \approx 0.567143.$

$W_0(-1) \approx -0.31813 + 1.33723i.$

$W_0(-\pi/2) = i\pi/2.$

$W_0(x^{x+1} \ln x) = x \ln x.$

$W_0(0) = W(0 \cdot e^0) = 0.$

$$W_0(z \ln z) = \ln z \quad \text{where} \quad (x \geq \frac{1}{e} \approx 0.36788)\,. \tag{3}$$

2 Extrication

Extrication is the name I'm giving to the process of replacing an expression that contains W with an expression that does not. There are two ways to do this:

In some very special cases, one can extract out of $W(\cdot)$ if the dot is one of a few numbers like those we saw above: 0, 1, e, etc, in which case the expression is replaced by a number of some sort.

The other way can occur under very special cases of an expression in x, say, such as xe^x, $x^{x+1}\ln x$, or $x\ln x$, which has the extrications:

$$W_0(xe^x) = x\,, \tag{4}$$

$$W_0(x\ln x) = \ln x \quad \text{where} \quad (x \geq \frac{1}{e} \approx 0.36788)\,, \tag{5}$$

$$W_0(x^{x+1}\ln x) = x\ln x\,. \tag{6}$$

3 Lemma 1: Changing Base

The basic problem I have when solving a problem that is 'almost in the correct form' to apply an extrication above, but not quite. Much time can be consumed performing a boring variable transformation that places the problem in the correct 'form'. Let me give an example. The following is problem 317 of Math Diversions that I solved in the Pset:

Given the relation

$$a + 3125^a = 0\,, \tag{7}$$

find the values of a. Now, if it were of the form $a + e^a = 0$, it would be a bit easier to solve, right? The following lemma let's us deal efficiently with (7).

Lemma 1:

Let s be a real number such that $s > 0$. Then, given

$$zs^z = B\,, \tag{8}$$

then

$$z = W_s(B)\,, \tag{9}$$

where

$$W_s(B) \equiv \frac{W(B\ln s)}{\ln s}\,, \tag{10}$$

which becomes the ordinary Lambert W function when $s = e$, which I refer to as the 'Lambert W function base s'.[1]

Proof:

Let

$$s^z = e^y\,, \tag{11}$$

and then take the logarithm:

$$z\ln s = y\,, \tag{12}$$

[1] This notation I invented myself.

and then solve for z,

$$z = \frac{y}{\ln s}\,. \tag{13}$$

Next, we substitute this stuff back into (8), to get

$$ye^y = B\ln s\,. \tag{14}$$

Now, when we take the Lambert W function across this equation, we have that

$$y = W(B\ln s)\,. \tag{15}$$

On returning to the variable z, we get

$$z = \frac{W(B\ln s)}{\ln s}\,. \tag{16}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(B\ln e)}{\ln e} = W(B)\,, \tag{17}$$

which is the usual Lambert W function.

Warning: Please don't get confused when you see me replace the base e with the base s in a calculation, as above. If you see W_0, W_{-1}, or W_n, these are the standard notations for various forms of the Lambert W function. In any case, my use of $W_s(B)$ is so non-standard, that I would (probably) only use it within an internal calculation, not as an end result.

Now, let's solve the problem above. We can begin by rewriting (7) as

$$-a = 3125^a\,, \tag{18}$$

or

$$-a\,3125^{-a} = 1\,. \tag{19}$$

We can now solve for $-a$:

$$-a = W_{3125}(1) = \frac{W(1\cdot\ln 3125)}{\ln 3125} = \frac{W(\ln 3125)}{\ln 3125}\,. \tag{20}$$

But $3125 = 5^5$, so

$$a = -\frac{W(5\ln 5)}{5\ln 5}\,. \tag{21}$$

If we want to be very general, we can write for the answer

$$a = -\frac{W_n(5\ln 5)}{5\ln 5} \quad \text{for} \quad n \in \mathbb{Z}\,, \tag{22}$$

which I shall not go into here. But for the principal value, which is (21), we can use another lemma (that we'll state and prove next) to simplify:

$$a = -\frac{W(5\ln 5)}{5\ln 5} = -\frac{\ln 5}{5\ln 5} = -\frac{1}{5}\,. \tag{23}$$

Clearly, using this lemma has saved us quite a few perfunctory steps.

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4 Lemma 2: Extricating $x\ln x$ from $W(\cdot)$

I'm referring to the principal value of W, given $W(x\ln x)$ as in (191). Anyway, if

$$x\ln x = B\,, \tag{24}$$

then

$$\ln x = W(B)\,. \tag{25}$$

Proof: Let $x = e^y$, then

$$W(x\ln x) \to W(e^y(y)) = W(ye^y) = y = \ln x\,. \tag{26}$$

Let's do an example problem.

Given the relation

$$x^x = e^{-\pi + i\ln 4}\,, \tag{27}$$

find the values of x.

We can begin by taking the natural logarithm across (27):

$$x\ln x = -\pi + i\ln 4 + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{28}$$

At first thought, it may seem as though applying the Lambert W function across this equation is taking a step backwards. I mean, what do we do with $W(x\ln x)$? But that's the beauty of it. Out of $W(x\ln x)$ we can extricate just $\ln x$.

Then we can apply the Lambert W function across this equation and use the lemma above, to get

$$\ln x = W(-\pi + i\ln 4 + 2\pi i n) \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{29}$$

Lastly, we just need to raise e to the power of Eq. (29),[2] to get

$$x = e^{W(-\pi + i\ln 4 + 2\pi i n)} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{30}$$

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[2] Yes, it sounds strange, but it works for me. The expression 'to take an object to the power of an equation' can be explained this way: Let our equation be 'LHS = RHS', and let our object be z. Then "z raised to the power 'LHS = RHS'" means the following:

$$z^{\text{LHS}} = z^{\text{RHS}}\,.$$

This is often particularly useful when dealing with logarithms.

5 Lemma 3: Extricating $x^{x+1}\ln x$ from $W(\cdot)$

The identity

$$W(x^{x+1}\ln x) = x\ln x \tag{31}$$

was in the list I pulled from Wikipedia. I list it here for completeness, though I have not used it so far.

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6 Lemma 4: $W(A)e^{W(A)} = A$

Let's begin with the relation

$$xe^x = A\,. \tag{32}$$

Then

$$x = W(xe^x) = W(A)\,. \tag{33}$$

If we now make the change of variable:

$$x = \ln y\,, \tag{34}$$

then (33) becomes

$$\ln y = W(y\ln y) = W(A)\,. \tag{35}$$

In other words, besides trying to conform the given expression to that of (32), we can alternatively conform it to this:

$$y\ln y = A\,, \tag{36}$$

Now, multiply (35) through by y:

$$y\ln y = yW(A)\,. \tag{37}$$

Applying the transitive property to these last two equations, yields

$$A = yW(A)\,. \tag{38}$$

But $y = e^x$, so then

$$A = e^x\,W(A) = e^{W(A)}\,W(A)\,, \tag{39}$$

where we used (33). Anyway, we have that

$$W(A)\,e^{W(A)} = A\,. \tag{40}$$

If my memory serves me correctly, I have used this identity to convert an answer that WolframAlpha got to compare it to the answer I got.

7 Sample Problems

Problem 191:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=elkUO6eRMvM
Title: Solving 8 Equations w/ Lambert W function
 Presenter: blackpenredpen
```

Out of the eight problems solved by the Presenter, I'm choosing to do number 6, but do it a bit differently.

Given the relation

$$W(e^{e^2+1+x^x}) = x^x\,, \tag{41}$$

find the values of x.

The Solution

For a warm-up on the Lambert W function, we need the following result:

$$W(ye^y) = y\,, \tag{42}$$

for $y \geq -1$.

Now, the Given equation can be rewritten as

$$W(e^{x^x}e^{e^2+1}) = x^x\,. \tag{43}$$

But according to (42), how are we supposed to get x^x as the output of the Lambert W function? This way (by 'pattern matching'):

$$W(x^x e^{x^x}) = x^x\,. \tag{44}$$

Therefore, we have to set

$$x^x = e^{e^2+1}\,. \tag{45}$$

Presenter continued by taking the logarithm of both sides. But I'll do it by the means I have typically approached this kind of problem — by a change of variable. First, we identify the 'base' value, which is 'e'. Next, we make the change of variable to

$$x = e^\alpha\,. \tag{46}$$

Then (45) becomes

$$(e^\alpha)^{(e^\alpha)} = e^{e^2+1}\,, \tag{47}$$

which simplifies to

$$e^{\alpha e^\alpha} = e^{e^2+1}\,. \tag{48}$$

On equating exponents, we have that

$$\alpha e^\alpha = e^2 + 1\,. \tag{49}$$

But this is begging us to use the Lambert W function to finish off this problem.

$$\alpha = W(\alpha e^{\alpha}) = W(e^2 + 1)\,. \tag{50}$$

Therefore

$$x = e^{W(e^2+1)}\,. \tag{51}$$

That was how I did it originally. These days, I do it a bit differently. Going back to (45), I would now take the logarithm, to get

$$x \ln x = e^2 + 1\,. \tag{52}$$

Next, I'll apply the Lambert W function, to get

$$\ln x = W(e^2 + 1)\,. \tag{53}$$

And then raise e to this equation to get[3]

$$x = e^{W(e^2+1)}\,. \tag{54}$$

Problem 262:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Fcp92u28eUY
Title: You Probably Haven't Seen This Before |
       Problem 212
 Presenter: aplusbi
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The Problem

Given the relation

$$\ln\left(\ln z\right) = z\,, \tag{55}$$

find the values of z.

A lemma I'll need is the following: If

$$y \ln y = B\,, \tag{56}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{57}$$

The Solution

[3]The expression 'to take an object to the power of an equation' can be explained this way: Let our equation be 'LHS = RHS', and let our object be z. Then "z raised to the power 'LHS = RHS'" means the following:

$$z^{\mathrm{LHS}} = z^{\mathrm{RHS}}\,.$$

My first goal is to unnest the logarithms and see what happens. So, let's raise the LHS and RHS as the exponents of e:

$$e^{\ln(\ln z)} = \ln z = e^z \,. \tag{58}$$

Now multiply through by z:

$$z \ln z = z e^z \,. \tag{59}$$

Now we take the Lambert W function across this equation.

$$\ln z = z \,, \tag{60}$$

where we used both the definition of the Lambert function and the above lemma. Next, we multiply through by $-z^{-1}$:

$$z^{-1} \ln z^{-1} = -1 \,. \tag{61}$$

Once again we take the W function.

$$\ln z^{-1} = W(-1) \,. \tag{62}$$

After a couple algebraic steps, we get:

$$z = e^{-W(-1)} \,. \tag{63}$$

Problem 293:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Q4MV_jqpH_k
Title: Cambridge University Admission Test Tricks !
 Presenter: Super Academy
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The Problem

Given the relation

$$5^{10x} = x^2 \,, \tag{64}$$

find the real values of x.

The Preparation

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$z a^z = B \,, \tag{65}$$

then

$$z = W_a(B) \,, \tag{66}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{67}$$

which becomes the ordinary Lambert W function when $a = e$.

I'll need the lemma:

$$W(y\ln y) = \ln y\,, \tag{68}$$

for the principal value of W and $y\ln y \geq -1/e$.

Proof: Let $y = e^w$, then

$$W(e^w(w)) = W(we^w) = w = \ln y\,. \tag{69}$$

The Solution

My first instinct is to take the square root on both sides, yielding,

$$5^{5x} = \pm x\,. \tag{70}$$

I tried to use the usual substitution of $x = 5^\alpha$, but I ran into unexpected trouble with that approach. My next approach is to use the Lambert W function. Of course, I just have to set it up first!

Dividing through by 5^{5x} and using some other algebra gives us:

$$\pm 1 = x(5^{-5})^x\,. \tag{71}$$

Now we take the Lambert W function base 5^{-5}, to get

$$x = W_{5^{-5}}(\pm 1) = \frac{W(\pm\ln 5^{-5})}{\ln 5^{-5}}\,. \tag{72}$$

Expanding,

$$x_\pm = \begin{cases} W_{5^{-5}}(+1) = \dfrac{W(+\ln 5^{-5})}{\ln 5^{-5}} = \dfrac{W(-5\ln 5)}{-5\ln 5}\,, \\ W_{5^{-5}}(-1) = \dfrac{W(-\ln 5^{-5})}{\ln 5^{-5}} = \dfrac{W(5\ln 5)}{-5\ln 5} = \dfrac{\ln 5}{-5\ln 5} = \dfrac{-1}{5}\,. \end{cases} \tag{73}$$

But only $x_- = -\frac{1}{5}$ will give a real value for x, and we can test it.

$$5^{10(-\frac{1}{5})} \stackrel{?}{=} (-\tfrac{1}{5})^2\,, \tag{74}$$

which is true.

Problem 299:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=glE7fMKJO1Q
Title: Oxford University Pure Mathematics Course Admission Exam
 Presenter: Super Academy
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8 The Problem

Given the relation

$$\log x + 64^{\log x} = \tfrac{1}{3}\,, \tag{75}$$

find the real values of x.

The Preparation

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{76}$$

then

$$z = W_a(B)\,, \tag{77}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{78}$$

which becomes the ordinary Lambert W function when $a = e$.

Another lemma I'll need is the identity (Lemma 2), that, if

$$y\ln y = B\,, \tag{79}$$

then

$$\ln y = W(y\ln y) = W(B)\,. \tag{80}$$

Thus,

$$W(y\ln y) = \ln y\,, \tag{81}$$

for the principal value of W and $y\ln y \geq -1/e$.

The Solution

Let's begin by introducing a new variable. Let

$$y = \log x - \tfrac{1}{3}\,, \tag{82}$$

so that the Given relation becomes

$$y = -64^{y+1/3} = -64^{1/3}\cdot 64^y = -4\cdot 64^y\,. \tag{83}$$

With a little algebra, we can rewrite this to

$$-y64^{-y} = 4\,. \tag{84}$$

On taking the Lambert W function base 64, we have that

$$\begin{aligned}-y = W_{64}(4) &= \frac{W(4\cdot\ln 64)}{\ln 64} = \frac{W(2^2\cdot\ln 2^{2\cdot 3})}{\ln 64}\\ &= \frac{W(2^3\cdot\ln 2^3)}{\ln 64} = \frac{\ln 2^3}{\ln 64} = \frac{3\ln 2}{6\ln 2} = \frac{1}{2}\,.\end{aligned} \tag{85}$$

Returning to $\log x$, we get

$$\log x = y + \tfrac{1}{3} = -\tfrac{1}{2} + \tfrac{1}{3} = -\tfrac{1}{6}\,. \tag{86}$$

Therefore,

$$x = 10^{-\frac{1}{6}} = 1/\sqrt[6]{10} = \frac{1}{\sqrt[6]{10}}\,. \tag{87}$$

Problem 313:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=GJbzsmccFtw
Title: all solutions to 2^x-3x-1=0 (transcendental equation)
 Presenter: blackpenredpen
```

The Problem

Given the relation

$$2^x - 3x - 1 = 0\,, \tag{88}$$

find the values of x.

The Preparation

The following is the 'Lambert W function base s'[4], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,. \tag{89}$$

Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{90}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{91}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

The Solution

Let's begin by stating the trivial solution $x = 0$. But can we find any others?

My first instinct was to use the Lambert W function. But before we can invoke the Lambert W function, we have to set up the relation for it. To make appearances easier, I want to introduce a temporary variable k, defined by

$$3k = 3x + 1, \quad \text{so} \quad x = k - \tfrac{1}{3}\,. \tag{92}$$

[4] This notation I invented myself.

With this, (88) becomes

$$2^{k-1/3} = 3k\,. \tag{93}$$

With a little algebra, this becomes

$$-k2^{-k} = -\frac{1}{3\sqrt[3]{2}}\,. \tag{94}$$

Applying the Lambert W function base 2, we get

$$-k = W_2\big(-\frac{1}{3\sqrt[3]{2}}\big) = \frac{W\Big(-\dfrac{\ln 2}{3\sqrt[3]{2}}\Big)}{\ln 2}\,. \tag{95}$$

Finally,

$$x = -\frac{1}{3} - \frac{W\Big(-\dfrac{\ln 2}{3\sqrt[3]{2}}\Big)}{\ln 2}\,. \tag{96}$$

Problem 314:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=0SDxMbrKh7E
Title:When Two Functions Are Tangent
 Presenter: SyberMath Shorts
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The Problem

Given the relation

$$e^x = \sqrt{ax}\,, \tag{97}$$

find the values of x.

The Preparation

The following is the 'Lambert W function base s'[5], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,. \tag{98}$$

Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{99}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{100}$$

[5]This notation I invented myself.

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

The Solution

Let's begin by squaring both sides:

$$(e^2)^x = ax\,. \tag{101}$$

With a little algebra, this becomes

$$-\frac{1}{a} = -x(e^2)^{-x}\,. \tag{102}$$

Hence, we can solve for $-x$:

$$-x = W_{e^2}\left(\frac{-1}{a}\right). \tag{103}$$

Lastly, we can solve for x:

$$x = -\frac{W\left(\frac{-1}{a}\ln e^2\right)}{\ln e^2} = -\frac{1}{2}W\left(\frac{-2}{a}\right). \tag{104}$$

Problem 331:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=R8rLyzJ9IK4
Title: How to solve this?
 Presenter: Super Academy
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The Problem

Given the relation

$$x^{x^8} = 8\,, \tag{105}$$

find the real, positive values of x.

The Preparation

I'll need the following lemma:

$$W(y\ln y) = \ln y\,, \tag{106}$$

for the principal value of W and $y\ln y \geq -1/e$.

Proof: Let $y = e^w$, then

$$W(e^w(w)) = W(we^w) = w = \ln y\,. \tag{107}$$

The Solution

I can think of two ways to solve this problem: The first is to perform a change of variable, making $x = 8^\alpha$, and taking it from there. The second is to use the Lambert W function, which is the route I'll take here.

So, let's begin by taking the natural logarithm across the Given relation, to get

$$x^8 \ln x = \ln 8\,. \tag{108}$$

Next, multiply through by 8:

$$8x^8 \ln x = 8 \ln 8\,, \tag{109}$$

or

$$x^8 \ln x^8 = 8 \ln 8\,. \tag{110}$$

Now, take the Lambert W function across this equation, to get

$$\ln x^8 = \ln 8\,, \tag{111}$$

where we used the above lemma. From this, we get that

$$x^8 = 8\,. \tag{112}$$

And finally,

$$x = 8^{1/8}\,. \tag{113}$$

Problem 349:

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Source: https://www.youtube.com/watch?v=RQYwPdz9RWI
Title: Simple trigonometric equation
 Presenter: Tambuwal Maths Class
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The Problem

Given the relation

$$\sin x = 4^{-\sin x}\,, \tag{114}$$

find the values of x.

The Preparation

To continue, I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{115}$$

then

$$z = W_a(B)\,, \tag{116}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{117}$$

which becomes the ordinary Lambert W function when $a = e$.

A lemma I'll need is the following: If

$$y\ln y = B\,, \tag{118}$$

then

$$\ln y = W(y\ln y) = W(B)\,. \tag{119}$$

The Solution

There are other ways to solve this problem than by using the Lambert W function, but this method has a generality to it.

Let's begin by bringing the two sides together as factors:

$$(\sin x)4^{\sin x} = 1\,. \tag{120}$$

Next, we take the Lambert W function base 4 across the equation:

$$\sin x = W_4(1) = \frac{W(1\cdot\ln 4)}{\ln 4} = \frac{W(2\ln 2)}{2\ln 2} = \frac{\ln 2}{2\ln 2} = \frac{1}{2}\,, \tag{121}$$

where we used the second lemma this time. Hence,

$$x = \begin{cases} \dfrac{\pi}{6} + 2\pi n & \text{for} \quad n \in \mathbb{Z}\,, \\ \dfrac{5\pi}{6} + 2\pi m & \text{for} \quad m \in \mathbb{Z}\,. \end{cases} \tag{122}$$

Problem 373:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=g8KWDPCXzCg
Title: An Interesting Exponential Equation
 Presenter: SyberMath
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The Problem

Given the relation

$$2x^{2x} = 1\,, \tag{123}$$

find the real values of x.

The Preparation

I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{124}$$

for the principal value of W and $y \ln y \geq -1/e$.

The Solution

We begin by dividing through by 2 and then taking the squareroot across the equation:

$$x^x = \pm\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}^{1/2}\,, \tag{125}$$

where we threw out the negative root. Next, we take the natural logarithm across this equation.

$$x \ln x = \ln{(\tfrac{1}{2})^{1/2}} = \tfrac{1}{2} \ln \tfrac{1}{2}\,. \tag{126}$$

On taking the Lambert W function across this equation and using the lemma above, we get

$$\ln x = \ln \tfrac{1}{2}\,. \tag{127}$$

From this we conclude that

$$x = \tfrac{1}{2}\,. \tag{128}$$

WolframAlpha also gives as a real solution

$$x = e^{W_{-1}(-\ln{(2)}/2)}\,, \tag{129}$$

which (seems) to come from that negative root we threw out in (125).

Problem 379:

The YouTube video is found at:

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Source: https://www.youtube.com/shorts/oxdDcIjdGwE
Title: Can you solve this ‘quite tough’ equation?
 Presenter: Gretsy Academy
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The Problem

Given the relation

$$W(x) = \ln{(5x)}\,, \tag{130}$$

find the values of x.

[Skip down to the solution, if you prefer.]

The Preparation

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{131}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{132}$$

The Solution

I know of two ways to extricate oneself from inside a Lambert W function. The first is by its definition:

$$W(ye^y) = y\,, \tag{133}$$

and the second is by this alternate

$$W(y \ln y) = \ln y\,. \tag{134}$$

Let's use this second way. Let

$$x = y \ln y\,. \tag{135}$$

Then (64) becomes

$$\ln y = \ln\left(5y \ln y\right). \tag{136}$$

After dropping the logarithms, we get

$$y = 5y \ln y\,. \tag{137}$$

Canceling the y's, we can solve for y:

$$y = e^{1/5}\,. \tag{138}$$

And finally,

$$x = \tfrac{1}{5} e^{1/5}\,. \tag{139}$$

Problem 384:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=ca_BOHCmRbo
Title: An Imaginarily Exponential Equation
     | Problem 386
 Presenter: aplusbi
```

The Problem

Given the relations

$$z = i^z \,, \tag{140}$$

find the complex values of z.

(Skip down to the solution, if you like.)

The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{141}$$

then

$$z = W(B) \,, \tag{142}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

The following is the 'Lambert W function base s'[6], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A \,. \tag{143}$$

Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s} \,. \tag{144}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A) \,, \tag{145}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

The Solution

In the lemma above, I stipulated that the base should be a real number greater than zero. This is a reasonable assumption, but I intend to be playful right now and let the base be i. Let's see what happens and let the Given relation be rewritten as

$$-zi^{-z} = -1 \,, \tag{146}$$

in anticipation of introducing the Lambert W function. Therefore,

$$-z = W_i(-1) = \frac{W(-1 \cdot \ln i)}{\ln i} \,. \tag{147}$$

But $i = e^{i\pi/2}$, so

$$-z = \frac{W(-i\pi/2)}{i\pi/2} \,. \tag{148}$$

[6]This notation I invented myself.

And finally,

$$z = \frac{2i}{\pi} W(-i\pi/2)\,. \tag{149}$$

Math Diversion Problem 386

Problem 386:

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=wmAYuYDFjE0
Title: You Should Learn This Trick
 Presenter: BriTheMathGuy
```

The Problem

Given the relation

$$x^{x^3} = 36\,, \tag{150}$$

find the real values of x.

(Skip down to the solution, if you like.)

The Preparation

Lemma 1: I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{151}$$

for the principal value of W and $y \ln y \geq -1/e$.

The Solution

Let's begin by cubing both sides of the Given relation:

$$(x^3)^{x^3} = 36^3\,. \tag{152}$$

Next, we take the natural logarithm.

$$(x^3) \ln (x^3) = 3 \ln 36\,. \tag{153}$$

Now we take the Lambert W function across this last equation, to get

$$\ln (x^3) = W(3 \ln 36) = W(6 \ln 6) = \ln 6\,. \tag{154}$$

Next, we raise e to this equation, to get[7]

$$x^3 = 6\,, \tag{155}$$

[7] To raise a number b to the 'power of an equation' simply means this: If the equation is 'LHS = RHS', then $b^{\mathrm{LHS}=\mathrm{RHS}}$ means $b^{\mathrm{LHS}} = b^{\mathrm{RHS}}$.

of which the real root is

$$x = 6^{1/3} . \tag{156}$$

Math Diversion Problem 387

Problem 387:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=9R0qz6_IKgw
Title: Harvard University Admission Interview Tricks
 Presenter: Super Academy
```

The Problem

Given the relation

$$3^{x-2} = x\,, \tag{157}$$

find the real values of x.

(Skip down to the solution, if you like.)

The Preparation

The following is the 'Lambert W function base s'[8], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,. \tag{158}$$

Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{159}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{160}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

The Solution

Let's begin by rewriting the Given relation to the following form:

$$3^{-2} = \frac{1}{9} = x3^{-x}\,, \tag{161}$$

[8]This notation I invented myself.

which can be rewritten into

$$-x3^{-x} = -\frac{1}{9}\,. \tag{162}$$

Now we take the Lambert W function base 3 across this last equation, to get

$$-x = W_3\big(-\tfrac{1}{9}\big) = \frac{W(-\frac{1}{9}\ln 3)}{\ln 3}\,, \tag{163}$$

or

$$x = -\frac{W(-\frac{1}{9}\ln 3)}{\ln 3}\,. \tag{164}$$

Now, we look at the details.

$$-\frac{1}{9}\ln 3 \approx -0.122068\,, \tag{165}$$

but

$$-\frac{1}{e} \approx -.367894\,, \tag{166}$$

Therefore,

$$-\frac{1}{e} \le -\frac{1}{9}\ln 3 < 0\,, \tag{167}$$

which puts $W(-\frac{1}{9}\ln 3)$ in the state of having two possible real values,

$$x = \begin{cases} -\frac{W_{-1}(-0.122068)}{\ln 3} & \approx 3.00000\,, \\ -\frac{W_0(-0.122068)}{\ln 3} & \approx 0.127869\,, \end{cases} \tag{168}$$

whose values I let WoframAlpha calculate. By the way, the integer 3 is clearly a solution to (157)

Math Diversion Problem 392

Problem 392:

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Owzej81Y5EA
Title: The DISTURBING TRUTH about Lambert W
 Presenter: Owls Math
```

The Problem

Given the relation

$$1 + x = x\ln\frac{1}{x}\,, \tag{169}$$

find the values of x.

(Skip down to the solution, if you like.)

The Preparation

Lemma 1: I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{170}$$

for the principal value of W and $y \ln y \geq -1/e$.

The Solution

Let's begin by rewriting (169) into the form

$$1 + x = -x \ln x\,. \tag{171}$$

Next, let's raise e to the power of Eq. (171).[9]

$$e^{1+x} = e^{-x \ln x} = \left(e^{\ln x}\right)^{-x} = x^{-x}\,, \tag{172}$$

which can be rewritten as

$$e = x^{-x} e^{-x} = (xe)^{-x}\,. \tag{173}$$

This can be further taken to

$$(xe)^x = e^{-1}\,. \tag{174}$$

Now raise both sides to the e power:

$$(xe)^{xe} = e^{-e}\,. \tag{175}$$

Next, take the logarithm:

$$(xe) \ln(xe) = -e\,. \tag{176}$$

Then the Lambert W function:[10]

$$\ln\,(xe) = W(-e)\,. \tag{177}$$

Then raise e to this equation:

$$xe = e^{W(-e)}\,. \tag{178}$$

And finally,

$$x = e^{W(-e)-1}\,. \tag{179}$$

WolframAlpha claims that this is two solutions, namely

$$x = \begin{cases} e^{W_0(-e)-1}\,, \\ e^{W_{-1}(-e)-1}\,. \end{cases} \tag{180}$$

I presume that these are nonreal solutions, first because WolframAlpha did not claim that they are real, and second because the domain for real solutions to $W(x)$ is for $-\frac{1}{e} \leq x < 0$, which makes the presented solutions out of bounds.

[9]To raise a number b to the 'power of an equation' simply means this: If the equation is 'LHS = RHS', then $b^{\mathrm{LHS=RHS}}$ means $b^{\mathrm{LHS}} = b^{\mathrm{RHS}}$.

[10]There are many ways to get from (171) to this equation.

9 Supplemental Problem#1

Given the relation

$$x = 2^{x-2}\,, \tag{181}$$

find the two real values of x.

Note: See my write-up on the Lambert W function for the Lambert W function base B.

10 Solution

Right off the bat, by inspection, I see that $x = 4$ is a real solution.

Anyway, this looks like a job for the Lambert W function. My first step is to transform (181) into

$$x = 2^x \frac{1}{4}\,. \tag{182}$$

Next, to

$$x2^{-x} = \frac{1}{4}\,. \tag{183}$$

And now to

$$(-x)2^{-x} = -\frac{1}{4}\,. \tag{184}$$

Now I take the Lambert W function base 2 across this equation, to get

$$-x = W_{(2)}(-\tfrac{1}{4}) = \frac{W_n(-\frac{1}{4}\ln 2)}{\ln 2} \quad \text{for } n \in \mathbb{Z}\,. \tag{185}$$

Thus, I get for the general solution

$$x = -\frac{W_n(-\frac{1}{4}\ln 2)}{\ln 2} \quad n \in \mathbb{Z}\,, \tag{186}$$

as also does WolframAlpha. Using some standard algebra, this becomes

$$x = \frac{W_n(\frac{1}{4}\ln\frac{1}{2})}{\ln\frac{1}{2}} \quad n \in \mathbb{Z}\,. \tag{187}$$

So, to use this formula to get the two real solutions (which WolframAlpha claims to be $x = 4$ and $x \approx 0.309907$), I have to assume that they follow from

$$x_{-1,0} = \frac{W_{-1,0}(\frac{1}{4}\ln\frac{1}{2})}{\ln\frac{1}{2}} \quad n \in \mathbb{Z}\,. \tag{188}$$

So, to get the value $x = 4$, I need to extricate out of the Lambert W function. I assume that means the principal value, given by

$$x_0 = \frac{W_0(\frac{1}{4}\ln\frac{1}{2})}{\ln\frac{1}{2}} \quad n \in \mathbb{Z}\,. \tag{189}$$

But I only have the following three common "extrication" formulas that seem appropriate to try:

$$W_0(x^{x+1}\ln x) = x\ln x\,, \tag{190}$$
$$W_0(x\ln x) = \ln x\,, \tag{191}$$
$$W_0(xe^x) = x\,. \tag{192}$$

But I couldn't figure out how to make any of them work.

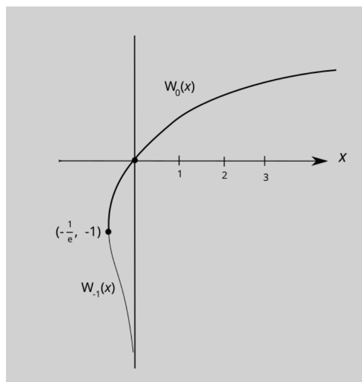

Figure 1. A sketch of the Lambert W function near the origin.

From the figure, we can see that the Lambert W function can yield real values on negative values between 0 and $-\frac{1}{e}$. W_0 is called the *principal* value of Lambert.

Let's go back to (186). To get the value $x = 4$, we need to have[11]

$$\frac{W_{-1}(-\frac{1}{4}\ln 2)}{\ln 2} = -4\,. \tag{193}$$

So, this must be correct, but how to prove it's correct? One way would be to use the approximation formulas to calculate it directly. That's not a very appealing method, when one knows the answer is an integer. The usual method I have employed successfully in the past has been to use one of the extrication formulas above. But as I said, I couldn't figure how to accomplish that this time.

So, I turned to outside help on this one. I explained the situation to this point to Copilot. Apparently Copilot could not succeed by using one of the extrication formulas either, so it tried the following method instead.

Consider the identity

$$W(x)e^{W(x)} = x\,. \tag{194}$$

[11]We won't get -4 using the principal value W_0 because it ranges only from -1 up to 0.

First, it converted the form to

$$We^W = -\tfrac{1}{4}\ln 2\,, \tag{195}$$

and tried the ansatz

$$W = -a\ln 2\,, \tag{196}$$

where a is to be determined. So let's substitute in:

$$(-a\ln 2)e^{(-a\ln 2)} = -\tfrac{1}{4}\ln 2\,, \tag{197}$$

which simplifies to

$$(-a\ln 2)2^{-a} = -\tfrac{1}{4}\ln 2\,, \tag{198}$$

which, after a bit of algebra, becomes

$$a = 2^{a-2}\,. \tag{199}$$

But this is just (181) all over again. ;)

So, in the end, neither Copilot nor I found a slick way to find the solution $x = 4$. By the way, Copilot agreed with WolframAlpha on the other real solution $x_0 \approx 0.309907$.