Selected Problems from *Elementary Linear Algebra* by Howard Anton

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Abstract

Herein are selected problems and their solutions from *Elementary Linear Algebra, 2ed* by Howard Anton.[1] I'll present here mostly solutions to problems, but not much of the theory that goes with it.

1 Chapter One

Problem5. Solve each of the following systems by Gaussian-Jordan elimination (p. 17).[2]

(a)

$$\begin{array}{ccccccc} 2x_1 & + & x_2 & + & x_3 & = & 8 \\ 3x_1 & - & 2x_2 & - & x_3 & = & 1 \\ 4x_1 & - & 7x_2 & + & 3x_3 & = & 10 \end{array} \tag{1}$$

First, we convert the set of simultaneous equations into an augmented matrix form. The goals are 1) to replace the nonzero entries of the bottom left of the matrix with zeros, and 2) replace the entries on the main diagonal by 1's, though this isn't imperitive to do this until the last steps. (In some problems, waiting could reduce the number of fractions to deal with.) Thus,

$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 8 \\ 3 & -2 & -1 & 1 \\ 4 & -7 & 3 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & -2 & -7 \\ 1 & -5 & 4 & 9 \\ 0 & -9 & 1 & -6 \end{array}\right] \begin{array}{l} r_2 - r_1 \\ r_3 - r_2 \\ r_3 - 2r_1 \end{array}$$

I've recorded the specific row operations I've performed along the way. For example, the expression $r_2 - r_1$ means that the first row has been replaced by the result of subtracting the second row from the first row. It's important to finish off a column before moving to the next column. Therefore, we need to put a zero in the first column, second row.

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & -2 & -7 \\ 0 & -2 & 6 & 16 \\ 0 & -9 & 1 & -6 \end{array}\right] r_2 - r_1 \longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & -2 & -7 \\ 0 & 1 & -3 & -8 \\ 0 & -9 & 1 & -6 \end{array}\right] r_2/(-2)$$

I set the pivot in the second row, second column to unity because it was easy to do without introducing fractions (or introducing *more* fractions as the case may be).

[1] *Elementary Linear Algebra, 2ed* by Howard Anton, Wiley, New York, 1977.
[2] There are three problems listed here but I'll only do the first two.

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -3 & -2 & -7 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & -26 & -78 \end{array}\right] 9r_2 + r_3 \qquad \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & -7 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right] r_3/(-26)$$

Okay, for the purpose of Gaussian elimination we're finish. What's left to do now is to perform 'back substitution'. So, let's do it. We begin on the bottom row. Clearly, we obtained $x_3 = 3$. Then, using this value in the second row, we get

$$x_2 - 3x_3 = -8\,, \tag{2}$$

or

$$x_2 - 3(3) = -8\,, \tag{3}$$

leaving us with

$$x_2 = 1\,. \tag{4}$$

Now there's only x_1 to calculate. Starting with

$$x_1 - 3x_2 - 2x_3 = -7\,, \tag{5}$$

we get

$$x_1 - 3(1) - 2(3) = -7\,, \tag{6}$$

which gives us

$$x_1 = 2\,. \tag{7}$$

Putting them together, we have that

$$x_1 = 2\,, \qquad x_2 = 1\,, \qquad x_3 = 3\,. \tag{8}$$

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5. (b)

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ -2x_1 & + & 5x_2 & + & 2x_3 & = & 0 \\ -7x_1 & + & 7x_2 & + & x_3 & = & 0 \end{array} \tag{9}$$

As before, we convert the set of simultaneous equations into an augmented matrix form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 0 \\ -7 & 7 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 14 & 8 & 0 \end{array}\right] \begin{array}{l} \\ 2r_1 + r_2 \\ 7r_1 + r_3 \end{array}$$

Then,

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] 2r_2 - r_3 \qquad \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] r_2/7$$

Let's go one more step:

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/7 & 0 \\ 0 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1 - r_2 \\ \\ \\ \end{array}$$

Okay, since we have found the system to have effectively one fewer equations than unknowns, we must solve for two of them in terms of the third. To do that, let's set $x_3 = t$, then $x_2 = -\frac{4}{7}t$, and $x_1 = -\frac{3}{7}t$. This solution is in the form of a parameteric equation of a line through the origin.

$$\mathbf{x} = \begin{pmatrix} -3/7 \\ -4/7 \\ 1 \end{pmatrix} t\,, \tag{10}$$

where t ranges over all real numbers. Hence, we get the same solution by replacing t by $-7t'$, yielding

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} t'\,, \tag{11}$$

where t' also ranges over all real numbers.

Problem 12. For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{array}{ccccccc} x & + & 2y & + & -3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array} \tag{12}$$

As before, we convert the set of simultaneous equations into an augmented matrix form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right] \begin{array}{l} \\ r_2 - 3r_1 \\ r_3 - 4r_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right] \begin{array}{l} \\ \\ r_3 - r_2 \end{array}$$

Analysis:

$$\begin{array}{lccl} \text{Case I} & : & a = +4 & \text{infinite solutions} \\ \text{Case II} & : & a = -4 & \text{no solutions} \\ \text{Case III} & : & |\,a\,|^2 \neq 4 & \text{one solution} \end{array} .$$

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Problem 3. Solve the following matrix equation for a, b, c, and d,

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix} . \tag{13}$$

Solution 3. Since we are solving for four unknowns, let's restructure (13) to get

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 2 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 7 \\ 6 \end{bmatrix} . \tag{14}$$

Now, we convert to augmented form and use elimination.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 2 & 0 & 0 & -4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 2 & 0 & -4 & -10 \end{array}\right] \quad r_4 - 2r_1$$

$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & -2 & -4 & -12 \end{array}\right] \quad r_4 - 2r_2 \qquad \longrightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right] \quad r_4 + 2r_3$$

$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad r_4/2 \qquad \longrightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad r_3 - 3r_4$$

$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad r_2 - r_3 \qquad \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad r_1 + r_2$$

Therefore, we have

$$a = 5 \quad b = -3, \quad c = 4, \quad d = 1\,. \tag{15}$$

Page 29, Problem 11. Show that the product of two diagonal matrices is diagonal.

Solution to Problem 11. This is easy to prove once you have a formula to multiply any two square matrices. Let A and B be two square matrices of dimension $n \times n$. First, know that the product of two $n \times n$ matrices is another $n \times n$ matrix. Let's define the product of these two matrices as M:

$$M \equiv AB\,. \tag{16}$$

Second, the ijth component of M, or M_{ij}, is given by the multiplication rule:

$$\begin{aligned} M_{ij} &= [\text{row } i \text{ of } A][\text{column } j \text{ of } B]^{\mathrm{T}} \\ &= [\,A_{i1}, A_{i2}, \ldots, A_{in}\,][B_{1j}, B_{2j}, \ldots, B_{nj}\,]^{\mathrm{T}} \\ &= A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{in}B_{nj}\,. \end{aligned} \tag{17}$$

Now, technically, all we need to do is to show that a given matrix is diagonal is to show that all of its off-diagonal elements are zero.

A random element of M below the main diagonal satsfies the constraint on the indices $i > j$ for i the row and j the column. But with so many zeros as the entries of both the row and column vector, the only product that is nonzero in the sum is when all the indices are the same. Such a product would look like $A_{kk}B_{kk}$. But there is no index k that can satisfy the index constraint we've imposed.

Hence the entries below the main diagonal are zero. And by a similar argument, the entries above the main diagonal are also zero. Thus, we have proved that the product of two diagonal square matrices is a diagonal matrix.

Page 38, Problem 19. Let $AX = B$ be any consistent system of linear equations, and let X_1 be a fixed solution. Show that every solution to the system can be written in the form

$$X = X_1 + X_0\,, \tag{18}$$

where X_0 is a solution to

$$AX = 0\,. \tag{19}$$

Show also that every matrix of this form is a solution. (I don't know what this means.)

Solution to Problem 19. This harmless looking problem hides a profound truth that you will likely run into again in other math subjects where linear systems occur. A system of the type found in (19) is said to be *homogeneous* and its solution X_0 is said to be the *homogeneous solution.* The solution to $AX = B$ is called the *particular solution* for it depends on the particular form of the matrix B. That X in (18) is also a solution of $AX = B$ is proved just by substitution into it. Hence,

$$AX = A(X_1 + X_0) = AX_1 + AX_0 = B + 0 = B\,. \tag{20}$$

By the way, X in (18) is called the *general solution.*

Page 45, Problem 6d. Find the inverse of the following matrix using elementary row operations.

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

Page 45, Solution to 6d.

First things first. What's the theory behind this? Given a presumably invertible $n \times n$ matrix A, we'll begin with the following steup:

$$[\,A\,|\,I\,]\,, \tag{21}$$

where I is the $n \times n$ unit matrix. Next, we'll perform any allowable elementary row operation E_1 on both sides of the vertical bar, as such

$$[\,E_1A\,|\,E_1I\,] = [\,E_1A\,|\,E_1\,]\,, \tag{22}$$

with the goal of reducing A one set closer to the unit identity matrix. Now, we'll repeat this process until the LHS of the bracketed quantity has become the identity, that is

$$[\,(\prod_{i=1}^{k} E_i)A\,|\,\prod_{i=1}^{k} E_i\,] = [\,I\,|\,\prod_{i=1}^{k} E_i\,]\,. \tag{23}$$

where k is some positive integer. We're almost finished.

We have that

$$(\prod_{i=1}^{k} E_i)A = I\,. \tag{24}$$

So, now just multiply by A^{-1} on the right on both sides:

$$\prod_{i=1}^{k} E_i = A^{-1}\,. \tag{25}$$

But $\prod_{i=1}^{k} E_i$ is what we arrived at on the RHS of the bracketed quantity in (23).

So, let's replace A above by $[\,A\,|\,I\,]$:

$$[\,A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \\ r_2 - r_1 \\ r_3 - r_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} \\ \\ r_3 - r_2 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} r_1 - 6r_3 \\ \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} r_1 - 6r_2 \\ \\ \\ \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} r_1/2 \\ \\ \\ \end{array}$$

Therefore,

$$A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Page 46, Problem 11. Express the matrix .

$$A = \begin{bmatrix} 1 & 3 & 3 & 8 \\ -2 & -5 & 1 & -8 \\ 0 & 1 & 7 & 8 \end{bmatrix}$$

in the form $A = EFR$, where E and F are elementary matrices, and R is in row-echelon form.

$$\begin{bmatrix} 1 & 3 & 3 & 8 \\ -2 & -5 & 1 & -8 \\ 0 & 1 & 7 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 3 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 1 & 7 & 8 \end{bmatrix} \; 2r_1 + r_2$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 3 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ \\ -r_2 + r_3 \end{array}$$

Hence,

$$R = \begin{bmatrix} 1 & 3 & 3 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

But

$$R = F^{-1}E^{-1}A \tag{27}$$

So, the matrix E^{-1} is the elementary matrix corresponding to the first row operation performed on the matrix A. The first row operation we performed was to replace row 2 by $2r_1 + r_2$, leaving the first and third rows unchanged. Hence

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{28}$$

and from this we get that

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

The next matrix F^{-1} is the elementary matrix corresponding to the next row operation performed on the matrix $F^{-1}A$. This row operation has the effect of leaving the first and second rows unchanged, but replacing row 3 by row 3 subtracting row 2, Hence

$$F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \tag{30}$$

and from this we get that

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \tag{31}$$

Finally,

$$\begin{bmatrix} 1 & 3 & 3 & 8 \\ -2 & -5 & 1 & -8 \\ 0 & 1 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

Page 52, Problem 9. Consider the matrices

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(a) Show that equation $AX = X$ can be written as $(A - I)X = 0$ and use this result to solve $AX = X$ for X.
(b) Solve $AX = 4X$.

For part a) $AX = X$ can be rewritten as $AX = IX$, where I is the 3×3 unit matrix. Then, $(A - I)X = 0$. Written out, we get

$$(A - I) \sim \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -4 & -2 \end{bmatrix} \begin{matrix} \\ r_1 - r_2 \\ r_3 - 2r_1 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} \\ \\ (-1/2)r_3 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{matrix} r_1 - r_3 \\ \\ r_3 - 2r_2 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ (-1/3)r_3 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} r_1 - 2r_3 \\ r_2 - 2r_3 \\ \\ \end{matrix}$$

Now, we set

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{33}$$

So, the only possible solution this can have is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{34}$$

This is not a surprising result. We will find out later that the action of an invertible linear transformation can only send the zero vector of the domain to the zero vector of the codomain.

Part b) is left for the reader.

Page 53, Problem 14. Let $AX = 0$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system

$$A^k X = 0 \tag{35}$$

also has only the trivial solution.

Solution: This looks like a simple induction problem. We're already given the base case, that

$$AX = 0 \tag{36}$$

has only the solution $X = 0$. Now, we assume that (35) is true with only the trivial solution, and show that

$$A^{k+1} X = 0 \tag{37}$$

is true with only the trivial solution. Assume not. Then

$$A^{k+1} X = 0 \tag{38}$$

has a solution other than the trivial solution. Then

$$A^{k+1} X = A(A^k X) = 0\,. \tag{39}$$

So, we must have $A^k X = 0$, but with X nontrivial, which is a contradiction. Hence, (38) must have only the trivial solution. And we're done. Thus, by the Principle of Mathematical Induction, $A^k X = 0$ has only the trivial solution for all positive integers k.