Basic Geometric Algebra

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The generic name for that language is *Geometric Algebra* (GA)....I expound it here in sufficient detail to be useful in instruction and research and to provide an entreé to the published literature.

— David Hestenes
Oersted Medal Lecture 2002:
Am. J. Phys. 71, 104–121 (2003)

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1 Introduction

Geometric algebra generalizes and contains a great many older algebras that we already know and love, such as linear algebra, complex numbers, vector spaces, tensors, Euclidean geometry, projective geometry, and so on. This introduction to the algebra will be as general as I can make it (with positive signature vectors, i.e., $\mathbf{v}^2 \geq 0$) but demonstrating its use by examples mostly in just two dimensions.

The modern form of Geometric Algebra has been defined by physicist David Hestenes, who started to work on it in the 1960s, adapting it from works by Marcel Riesz and the established Clifford algebra, which are characterized as a matrix algebra. The Clifford algebras had been introduced into physics in the 1920s under the umbrella of quantum mechanics in the specific forms of the Pauli and Dirac algebras.

So, now we arrive at the first place where confusion can arise. I'm going to define an 'algebra' for our purposes in this paper. An algebra $\mathcal{A}$ begins with a linear system A: Thus the sum of any two elements of A is another element in A. And A also has the operation of *scalar multiplication* defined on it.

Typically, these scalars are elements of some common field, like the real or complex numbers. Hence, for each scalar λ and for each element $\mathbf{v}$ of the linear space A, then $\lambda\mathbf{v} \in A$. In A, addition is both associative and commutative. If you're familiar with the mathematician's definition of a vector space, this should appear much like that. And, in fact, it is. The problem with calling it a vector space is the word 'vector' itself, because it's used in so many different ways, especially to physicists. And like the vector space, A will have a dimension n, a positive integer. (Because this is a basic introduction to GA, we will consider only finite-dimensional linear (vector) spaces at this time.)

And here's a defining point about the geometric algebra: Its scalars are only ever the real numbers. This constraint is self-imposed to force the physicist to find the geometric interpretations in the vectors and multivectors, and not in the scalars, especially not in complex scalars. The Clifford algebras typically use complex scalars.

And then there is the odd ball "geometric algebra" built over a one-dimensional vector space. It does not use matrices. Just pick a single unit vector, say u, take the complex linear combinations of $\{1, u\}$ to form the algebra, which has been named the **Unipodal Algebra** and I have written extensively on it. By the way, this algebra is isomorphic to the **bicomplex algebra**, which you can look into if you wish.

Let's take a moment to list the axioms for a vector space.

In the list below, let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be arbitrary vectors in V, and a and b be scalars in $\mathcal{F}$.

Axiom	**Meaning**
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	$\exists$ an element $\mathbf{0} \in V$, called the zero vector, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$, $\forall\ \mathbf{v} \in V$.
Inverse elements of addition	$\forall\ \mathbf{v} \in V$, $\exists$ an element $-\mathbf{v} \in V$, called the additive inverse of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multi. with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$
Identity element of scalar multi.	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in $\mathcal{F}$.
Distributivity of scalar multipli. with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
Distributivity of scalar multipli. with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
Commutivity of scalar multipli.	$a\mathbf{v} = \mathbf{v}a$.

Table 1. Axioms for a vector space.

Our next step is a big one. We allow the elements of A to multiply each other, and in doing so generate the Geometric Algebra $\mathcal{A}$ constructed on A. (Notice the change in symbol going from the vector space to the algebra built on the vector space.) So, let A and B be any two elements in $\mathcal{A}$, then the algebra constructed as the extension of $\mathcal{A}$ is closed under the geometric product:

$$AB \in \mathcal{A}\,. \tag{1}$$

2 Formal GA

Let V be an n-dimensional vector space with real scalars. Then the geometric algebra built on V, called $\mathcal{V}$, is the collection of all sums of all products in the form of what are called **multivectors** M:

$$M = \sum_{k=1}^{n} \langle\, M \,\rangle_k\,, \tag{2}$$

which is an expansion by grade.

The geometric product is associative. So, for all multivector A, B, C,

$$A(BC) = (AB)C\,. \tag{3}$$

This is an axiom of the geometric algebra, but it is foreseeable in that the geometric algebra is a generalization of matrix algebra.

We can add to our list of axioms for the geometric algebra that for every multivector A,

$$1A = A\,. \tag{4}$$

Now, we're at another tricky part of the development. By axiom, the square of every vector is a scalar. For this introduction, we're dealing only with Euclidean spaces and so the square of every vector is a nonnegative real number,[1] the square of a vector being zero only if the vector is the zero vector.

$$\mathbf{a}^2 \geq 0\,. \tag{5}$$

Now, we slowly build up complexity of vector multiplications.

$$\mathbf{ab} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) + \tfrac{1}{2}(\mathbf{ab} - \mathbf{ba})\,. \tag{6}$$

If you take a close look at this last equation, you'll see that we haven't really done anything. Essentially, all we did was to add in and subtract out the quantity $\frac{1}{2}\mathbf{ba}$. So, we'll look at the two term individually. First $\frac{1}{2}(\mathbf{ab} + \mathbf{ba})$. We already have one constraint on the system to clue us in, and that is that

$$\mathbf{aa} \in \mathbb{R}\,. \tag{7}$$

[1]When geometric algebra is designed to deal with special relativity, it's called **spacetime algebra**, and it has both square positive and square negative vectors.

Hence

$$\mathbf{aa} = \tfrac{1}{2}(\mathbf{aa}+\mathbf{aa}) + \tfrac{1}{2}(\mathbf{aa}-\mathbf{aa}) = \tfrac{1}{2}(\mathbf{aa}+\mathbf{aa}) \in \mathbb{R}\,. \tag{8}$$

So, perhaps we should consider that the quantity $\mathbf{ab}+\mathbf{ba}$ is also a real number. In fact, we'll now make this assumption.

$$\tfrac{1}{2}(\mathbf{ab}+\mathbf{ba}) \equiv \mathbf{a}\cdot\mathbf{b} \in \mathbb{R}\,, \tag{9}$$

where we say this is "**a** dot **b**," the standard Gibbs's dot product when the vector space is three-dimensional. That leaves us with

$$\tfrac{1}{2}(\mathbf{ab}-\mathbf{ba}) \equiv \mathbf{a}\wedge\mathbf{b}\,, \tag{10}$$

where we say this is "**a** wedge **b**. Going back to (6), we can now rewrite it as

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\,. \tag{11}$$

Okay, so $\mathbf{a}\cdot\mathbf{b}$ is a scalar, and it's obvious that

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}\,. \tag{12}$$

Whereas from (10), we see that

$$\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}\,. \tag{13}$$

From the antisymmetry of the wedge product, we see that any vector wedge itself is zero. Soon, we'll see how that the geometric algebra interprets the wedge of two distinct vectors as a directed plane segment.

Every multivector can be expressed as the sum of its graded parts. For example, in (11), **ab** has a scalar part, and we can express that as

$$\langle\,\mathbf{ab}\,\rangle_0 = \mathbf{a}\cdot\mathbf{b}\,. \tag{14}$$

and it has a bivector part

$$\langle\,\mathbf{ab}\,\rangle_2 = \mathbf{a}\wedge\mathbf{b}\,. \tag{15}$$

We can put this together symbolically:

$$\mathbf{ab} = \langle\,\mathbf{ab}\,\rangle_0 + \langle\,\mathbf{ab}\,\rangle_2\,. \tag{16}$$

By the way, to express that **ab** has no vector part, we can write

$$\langle\,\mathbf{ab}\,\rangle_1 = 0\,. \tag{17}$$

In a finite-dimensional vector space of dimension n the highest number of vectors that can be wedged together before the term vanishes of necessity is n. If $\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$ forms a basis for a vector space of dimension n, then

$$\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_n \neq 0\,. \tag{18}$$

In other words, there is no nonzero multivector in this geometric algebra of grade $n+1$ or higher.

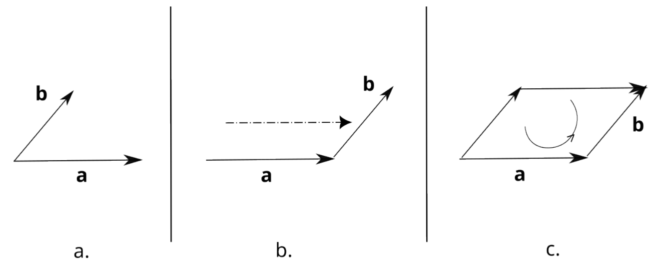

Figure 1. This graphic shows the formation of a bivector $\mathbf{a} \wedge \mathbf{b}$ by wedging two different vectors, $\mathbf{a}$ and $\mathbf{b}$, together. They start off with bases coincident and then the second vector 'slides' over the first until the base of the second meets the tip of the first. In Part c. we see the result being interpreted as an oriented parallelogram.

In Fig. 1 we see the formation of a bivector (an oriented plane segment) by having the vector $\mathbf{b}$ slide — parallel to itself — along vector $\mathbf{a}$. We can think of $\mathbf{a}$ as 'sweeping out' the parallelogram. However, shape is not the determining factor when comparing bivectors for equality. Two bivectors are equal if they have the same magnitude and orientation.

For a deeper study on this, see pages 20–30 of NFCM [1], which includes the generalization to $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$.

3 Reversion

We introduce now a highly convenient operation on a multivector M, called **reversion**, indicated by $M^\dagger$. Let's first check out what this operation does to scalars. Let α be a scalar, then

$$\alpha^\dagger = \alpha\,. \tag{19}$$

Next, we consider what reversion does to a geometric product of multivectors A and B.

$$(AB)^\dagger = B^\dagger A^\dagger\,. \tag{20}$$

If you're familiar with matrices, perhaps you can see a similarity with the transpose operation on matrix products.

Okay, so what happens when we apply the reversion operator to a simple r-vector, like $a_1 \wedge a_2 \ldots \wedge a_r$? We assume that reversion commutes with taking the grade operation and it distributes of graded parts of a multivector. So,

$$(a_1 \wedge a_2 \wedge \ldots \wedge a_r)^\dagger = \langle\, (a_1 a_2 \ldots a_r)^\dagger \,\rangle_r = \langle\, (a_r \ldots a_2 a_1)\,\rangle_r\,. \tag{21}$$

Therefore,

$$(a_1 \wedge a_2 \wedge \ldots \wedge a_r)^\dagger = a_r \wedge a_{r-1} \ldots \wedge a_1 \,. \tag{22}$$

So, how do we compare $a_1 \wedge a_2 \ldots \wedge a_r$ with $a_r \wedge a_{r-1} \wedge \ldots \wedge a_1$? Simple. We begin by repeatedly transposing the a_1 vector, currently at the right of $a_r \wedge a_{r-1} \wedge \ldots \wedge a_1$, with the vector to its immediate left, until it's on the left-most position, that is,

$$a_r \wedge a_{r-1} \wedge \ldots \wedge a_1 = (-1)^{r-1} a_1 \wedge a_r \wedge a_{r-1} \wedge \ldots \wedge a_2 \,, \tag{23}$$

where it has picked up a minus sign factor for each transposition. And then we do similarly with the a_2 vector, until its up against the a_1 factor. An so on. With each transposition, we pick up another factor of (-1), making for a total of $r-1$ of them. By the time we're finished with making the last transposition, we get

$$a_r \wedge a_{r-1} \wedge \ldots \wedge a_1 = (-1)^{r(r-1)/2} a_1 \wedge a_2 \wedge \ldots \wedge a_r \,. \tag{24}$$

We should already have an expectation when we apply the reversion operation to a simple bivector $a_i \wedge a_j$:

$$(a_i \wedge a_j)^\dagger = a_j \wedge a_i = -a_i \wedge a_j \,. \tag{25}$$

And for a vector $\mathbf{a}$, it's pretty obvious that

$$\mathbf{a}^\dagger = \mathbf{a} \,. \tag{26}$$

But this is just the simplest case of the invariance of a palindrome under the reversion operation. For $A = A_r$ and $B = B_s$, an r blade and an s blade, respectively, we have that:

$$(ABBA)^\dagger = ABBA \,. \tag{27}$$

It's to be pointed out that this might not be true if A and B are not both blades.[2] Under what more general conditions would (27) be true?

Now, we'll accept as axiomatic that for any two multivectors M and N,

$$(M+N)^\dagger = M^\dagger + N^\dagger \,. \tag{28}$$

Let $\mathcal{V}$ be a 3-d real vector space, and let $\mathcal{G}^3$ be the geometric algebra on $\mathcal{V}$. Further, let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three linearly independent vectors in $\mathcal{G}^3$. Then

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})^\dagger = \mathbf{c} \wedge \mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \,. \tag{29}$$

Let $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$ be a right-handed orthonormal set of basis vectors for $\mathcal{V}$. The wedge product of any two of these vectors, say the $\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j$ vectors, forms a

[2] Let's consider the case in which A and B are both simple blades as above, that is, they can be expressed as $A = a_1 \wedge a_2 \wedge \ldots \wedge a_s$ and $B = b_1 \wedge b_2 \wedge \ldots \wedge b_r$. Then (27) is true for $A^\dagger = \pm A$ and $B^\dagger = \pm B$.

bivector for the $\boldsymbol{\sigma}_i$-$\boldsymbol{\sigma}_j$ plane. The wedge product of all three of these forms a very special 3-vector in $\mathcal{G}^3$, named i, and called the **unit pseudoscalar** of $\mathcal{G}$.

$$i \equiv \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 \,. \tag{30}$$

From (29) we know that

$$i^\dagger = -i \quad \text{and} \quad i^2 = -1 \,. \tag{31}$$

We can use the unit pseudoscalar to define the Gibbs's cross product, using the wedge product.

$$\mathbf{a} \wedge \mathbf{b} = i\mathbf{a} \times \mathbf{b} \,. \tag{32}$$

So, yes, this is an implicit definition. We can be explicit by doing this:

$$\mathbf{a} \times \mathbf{b} \equiv -i\mathbf{a} \wedge \mathbf{b} \,. \tag{33}$$

4 Magnitudes

In Euclidean spaces every multivector has a magnitude. Let M be a multivector. Its magnitude $|M|$ is given by

$$|M|^2 = \langle M^\dagger M \rangle_0 = \langle M^\dagger M \rangle \,. \tag{34}$$

Since in physics, we are most familiar with the magnitudes of vectors, let's start with them. Let $\mathbf{a}$ be a nonzero vector in a geometric algebra, then its square magnitude is given by

$$|\mathbf{a}|^2 = \langle \mathbf{a}^\dagger \mathbf{a} \rangle_0 \,. \tag{35}$$

But $\mathbf{a}^\dagger = \mathbf{a}$, so that

$$|\mathbf{a}|^2 = \langle \mathbf{a}\mathbf{a} \rangle_0 = \mathbf{a}^2 \,, \tag{36}$$

where we have used the fact that the square of a vector must be a scalar. But $\mathbf{a}^2$ is a real number, which we'll name as α^2, so that

$$|\mathbf{a}|^2 = \alpha^2 \,. \tag{37}$$

Hence

$$|\mathbf{a}| = \alpha \,. \tag{38}$$

Now, if we divide through by α, we get

$$\left| \frac{\mathbf{a}}{\alpha} \right| = 1 \,. \tag{39}$$

So, what do we have? We have a vector $\mathbf{a}/\alpha$ whose magnitude is unity. We call such a vector a **unit vector**. For my purposes, a vector with a hat over it is a unit vector. Thus, given any nonzero vector $\mathbf{a}$, its unit vector is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \,. \tag{40}$$

Alternatively, we can write

$$\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}}\,, \tag{41}$$

or even

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{a}^2}\,. \tag{42}$$

Thus, every nonzero vector has a well-defined inverse.

5 Generalizing the Inner Product

Let $A_r = a_1 \wedge \ldots \wedge a_r$, then

$$a \cdot A_r \equiv \langle\,\mathbf{a}A_r\,\rangle_{r-1}\,, \tag{43}$$

$$a \wedge A_r \equiv \langle\,\mathbf{a}A_r\,\rangle_{r+1}\,, \tag{44}$$

To actually calculate an inner product such as (43), we have

$$a \cdot a_1 \wedge \ldots \wedge a_r = \sum_{k=1}^{r} (-1)^{k+1} a \cdot a_k\, a_1 \wedge \ldots \wedge \overset{\vee}{a_k} \wedge \ldots \wedge a_r\,, \tag{45}$$

where the vee over the vector means that it is omitted from the term.

> Thought for the day: What did the sassy sweet potato say to the philosopher? — I think, therefore I yam!

6 Second Rehashing

Since the time that Gibbs codified the so-called Gibbs vector algebra, which specially use the dot and cross products, to facilitate the use of the formulas in physics, such as work

$$\text{Work done} = \int \mathbf{F} \cdot d\mathbf{x}\,, \tag{46}$$

and angular momentum ℓ

$$\ell = \mathbf{r} \times \mathbf{p}\,, \tag{47}$$

people have not been unanimously happy with it. Although geometric algebra has its uses for the cross product, this is not usually the best place to use it. In celestial mechanics, for example, the vectors $\mathbf{r}$ and $\mathbf{p}$ often lie in a fixed plane in space, and thus to invoke the direction normal to this plane (which is along the direction $\mathbf{r} \times \mathbf{p}$) is superfluous, at least when using geometric algebra. We can instead write

$$\mathbf{L} = \mathbf{r} \wedge \mathbf{p}\,, \tag{48}$$

where $\mathbf{L}$ is a bivector.

Gibbs vector algebra violates one of the most fundamental notions of any formal algebraic system, namely, closure. That is, the operations on the elements of an algebra should always return values (elements, objects) in the same space. Gibbs vector algebra succeeds to do this in his vector cross product because it takes in two vectors of the space of vectors and returns another vector in the space. In the case of (47), the cross product takes in the vectors $\mathbf{r}$ and $\mathbf{p}$ and returns the vector $\mathbf{L}$.[3]

However, the dot product takes in two vectors and returns a scalar. Ironically, this is how it was defined from the getgo by

$$\mathbf{a} \cdot \mathbf{b} \equiv |\mathbf{a}|\,|\mathbf{b}| \cos\theta\,, \tag{49}$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the norms of the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, and θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$.

Back to the issue of vector-space closure, one could reply: 'What's so wrong with that? Didn't we send a man to the moon on Gibbs vector algebra?' Well, yes! The question is, however, can we keep the best of Gibbs vector algebra, yet improve the foundations of it? Yes again.

One way do that would be to assign an arbitrary direction in your vector space to send all scalar products to it. (You could even increase the dimensionality of your vector space to do this.) Ta-dah! It's now a 'vector'!

However, the mathematicians thought to fix the problem in a very different manner. They invented a new vector space – a companion space, called the *dual space*, to the original vector space $\mathcal{V}$. They gave this space a symbol $\mathcal{V}^*$ to denote it. And it works by having the vectors in the dual space (the dual vectors) act on the vectors in the original space to map them into scalars. Thus, Gibbs can keep his cross product, but his dot product has been replaced by the inner product between a vector space and its dual space.

But there is another solution to this problem suggested decades ago by physicist David Hestenes, which is to increase the size of your space so that scalar and vectors are both legitimate objects in the space.[4] How did he do this? By moving from the restrictions of the 'vector' space, as conceived of by Gibbs, to the linear space conceived of by Clifford.

Since it's not my purpose here to give a general overview of the many useful Clifford algebras in use today, I will restrict my presentation of them to that use in Hestenes's book *New Foundations for Classical Mechanics* [**?**], which uses the geometric (Clifford) algebra $\mathcal{G}^3$. This is an eight-dimensional linear space, in which every element of it can be expressed as a linear combination of a scalar, three vectors, three bivectors, and one pseudoscalar. By the way, the scalars in this algebra are the real numbers.

[3]It seems odd to me that two vectors in the same 'space' can have different units. I've never seen a mathematician or physicist clearly address this issue. There is one reference to it online at http://math.stackexchange.com/questions/864551/how-can-vectors-with-different-units-position-speed-share-the-same-spa. Perhaps if a mathematician were forced to address this question regarding the mutation of his or her precious unitless vector space, he or she might wince and say "Physicists!"

[4]As I stated before, Hestenes was not the first to promote geometric algebras in physics, but he has been the most outspoken and productive proponent for it.

This space is closed under addition and/or multiplication of any two elements of the algebra. And if the resulting dimensionality of a product is greater than three, the product is sent to zero.

Let's now construct $\mathcal{G}^3$ as a geometric algebra over a 3D vector space $\mathcal{V}^3$. First, we begin with our notion of scalar field, in this case the reals. Let α be a real number. Then we pick a 3-dimensional vector space, and all the rules of scalar multiplication and vector addition apply. Let A, B, C be any three elements of $\mathcal{G}^3$. Then we add to that a new form of product, called the *geometric product*, which is evident by juxtaposition on symbols. And here are some of the axiomatic rules just previously.

Generally speaking, multivectors do not commute with each other. The following are some axioms of the geometric algebra for E^n.[5]

$$A + B = B + A \tag{50a}$$
$$A(B + C) = AB + AC \tag{50b}$$
$$(B + C)A = BA + CA \tag{50c}$$
$$(AB)C = A(BC) \tag{50d}$$
$$\alpha(AB) = (\alpha A)B = A(\alpha B) = A(B\alpha) \tag{50e}$$
$$\mathbf{aa} = \mathbf{a}^2 = |\mathbf{a}|^2 \in \mathbb{R}^+ \quad \mathbf{a} \neq \mathbf{0} \tag{50f}$$
$$1M = M \tag{50g}$$
$$M = \sum_{k=1}^{n} \langle\, M \,\rangle_k \tag{50h}$$
$$\langle\, A + B \,\rangle_k = \langle\, A \,\rangle_k + \langle\, B \,\rangle_k \tag{50i}$$
$$A = B \implies \langle\, A \,\rangle_k = \langle\, B \,\rangle_k \tag{50j}$$

Table 2

To this list we can add that for every element of $\mathcal{G}^3$ (also referred to as a *multivector*), there exists an additive identity 0, a multiplicative identity 1, and an additive inverse: If M is a multivector, there exists is additive inverse $-M$, such that $M + (-M) = 0$.

From (50d), we see that the that geometric product is fully associative, which will greatly facilitate solving problems and presenting a multitude of identities. It is not generally the case that any two elements of this geometric algebra commute with each other, but it is the case that scalars commute with all multivectors of the algebra, and that any two elements of the algebra that differ only by being scalar multiples of each other commute. It's also the case that bivectors and trivectors each square to a scalar but it will be negative.

[5] We can instead just do geometric algebra in $\mathbb{R}^n$ by insisting on having a basis, but much can be done without introducing a basis.

Thus, they play roles in geometric algebra similar to the roles of imaginary numbers in the complex numbers, making formal imaginary numbers (i.e., being uninterpreted by use of elements in the geometric algebra) redundant.

Let $\mathbf{a}, \mathbf{b}$ be any two vectors of $\mathcal{G}^3$. Let's begin our investigation of this algebra by considering the general noncommutivity of vectors. The axioms of this algebra allow us to write the following identity:

$$\mathbf{ab} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) + \tfrac{1}{2}(\mathbf{ab} - \mathbf{ba})\,. \tag{51a}$$

For obvious reasons, we'll refer to $\frac{1}{2}(\mathbf{ab} + \mathbf{ba})$ as the symmetric part of $\mathbf{ab}$, and $\frac{1}{2}(\mathbf{ab} - \mathbf{ba})$ as the antisymmetric part of $\mathbf{ab}$. In the case that $\mathbf{a}$ and $\mathbf{b}$ are nonzero collinear vectors (scalar multiples of each other) the antisymmetric part of the product is zero but the symmetric part is nonzero. This is suggestive that the symmetric part is always a scalar. Now, let's switch the order of the vectors.

$$\mathbf{ba} = \tfrac{1}{2}(\mathbf{ba} + \mathbf{ab}) + \tfrac{1}{2}(\mathbf{ba} - \mathbf{ab})\,. \tag{51b}$$

Again, we see that if the vectors are collinear the symmetric part survives but the antisymmetric part does not.

But before we can fully justify insisting that the symmetric part is a scalar, we need one more axiom — that axiom of grade.

Every multivector M in $\mathcal{G}^3$ can be written as the sum of scalar, vector, bivector, and pseudoscalar parts, or M_0, M_1, M_2, and M_3, respectively, then the scalar, vector, bivector, and pseudoscalar parts can be expressed as

$$M = M_0 + M_1 + M_2 + M_3\,. \tag{52}$$

To facilitate dealing with grades, we define a grade operator $\langle\ \rangle_r$, which takes in an arbitrary multivector and extracts its rth part. That is,

$$\langle M \rangle_r = M_r\,. \tag{53}$$

Definition: A multivector K is said to be an r-vector if it's equal to its rth graded part, that is, if $K = \langle K \rangle_r$. For the scalar part of the multivector, we can write either $\langle M \rangle_0$ or $\langle M \rangle$, omitting the subscript.

Clearly,

$$M = \sum_{r=0}^{3} \langle M \rangle_r = \sum_{r=0}^{3} M_r\,. \tag{54}$$

We need to deal with the issue of existence of graded objects higher than 1. We have already specified that multivectors are made from sums and products of scalars and vector, but we have not yet proved that any thing of higher grade exists in the algebra. We do so now.

Axiom of Grade Splitting: Let $\mathbf{a}$ be any nonzero vector and let B be any nonzero r-vector other than a scalar. Then,[6]

$$\mathbf{a}B = \langle \mathbf{a}B \rangle_{r-1} + \langle \mathbf{a}B \rangle_{r+1}\,. \tag{55}$$

[6] For reference, see NFCM p. 36 [**?**].

Now let's go to cases. For B a pseudoscalar P $(r = 3)$

$$\mathbf{a}P = \langle\, \mathbf{a}P \,\rangle_2 + \langle\, \mathbf{a}P \,\rangle_4 = \langle\, \mathbf{a}P \,\rangle_2 \,, \tag{56}$$

where we dropped $\langle\, \mathbf{a}B \,\rangle_4$ since it's identically zero. For B a bivector $\mathbf{B}$ $(r = 2)$

$$\mathbf{aB} = \langle\, \mathbf{aB} \,\rangle_1 + \langle\, \mathbf{aB} \,\rangle_3 \,. \tag{57}$$

And lastly, for B a vector $\mathbf{b}$ $(r = 1)$,

$$\mathbf{ab} = \langle\, \mathbf{ab} \,\rangle + \langle\, \mathbf{ab} \,\rangle_2 \,, \tag{58}$$

where we dropped the subscript 0.

We need to note special handling for two special cases. Inner and outer products with a scalar. Let α be an arbitrary scalar and A_r be any r blade with $r \geq 1$. Then

$$\alpha \cdot A_r = 0 \,, \tag{59a}$$
$$\alpha \wedge A_r = \alpha A_r \,. \tag{59b}$$

Now we're in a position to interpret (51a).

Corollary of Grade Splitting:

$$\langle\, \mathbf{ab} \,\rangle = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) \,, \tag{60}$$
$$\langle\, \mathbf{ab} \,\rangle_2 = \tfrac{1}{2}(\mathbf{ab} - \mathbf{ba}) \,. \tag{61}$$

Now we present some fundamental definitions for products besides the geometric product. Let $\mathbf{a}$ and B as given in the Axiom of Grade Splitting above. Then the dot and wedge products are defined as

$$\mathbf{a} \cdot B = \langle\, \mathbf{a}B \,\rangle_{r-1} \,, \tag{62}$$
$$\mathbf{a} \wedge B = \langle\, \mathbf{a}B \,\rangle_{r+1} \,. \tag{63}$$

And then (55) can be replaced by

$$\mathbf{a}B = \mathbf{a} \cdot B + \mathbf{a} \wedge B \,. \tag{64}$$

And one more time back to (51a), we have that

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \,, \tag{65}$$

where

$$\mathbf{a} \cdot \mathbf{b} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) \quad \text{and} \quad \mathbf{a} \wedge \mathbf{b} = \tfrac{1}{2}(\mathbf{ab} - \mathbf{ba}) \,. \tag{66}$$

It's not difficult to show that our GA dot product distributes over addition. Let's do that now.[7]

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \tfrac{1}{2}[\mathbf{a}(\mathbf{b}+\mathbf{c})] + \tfrac{1}{2}[(\mathbf{b}+\mathbf{c})\mathbf{a}] \\ &= \tfrac{1}{2}[\mathbf{ab}+\mathbf{ac}] + \tfrac{1}{2}[\mathbf{ba}+\mathbf{ca}] \\ &= \tfrac{1}{2}[\mathbf{ab}+\mathbf{ba}] + \tfrac{1}{2}[\mathbf{ac}+\mathbf{ca}] \\ &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}\,. \end{aligned} \tag{67}$$

This result can be inductively extended to prove that the dot product distributes over any finite number of summed terms.

Our next task is to consider what happens to (49) when **a** and **b** are orthogonal to each other. In that case, $\theta = 90°$, cos $\theta = 0$, and $\mathbf{a}\cdot\mathbf{b} = 0$. Therefore we have to force this constraint on the GA dot product by this means:

Lastly, we need to add the physical constraint that the dot product of two unit vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ is the cosine of the angle θ between them. This then adds the constraint to the system

$$\cos\theta = \tfrac{1}{2}(\hat{\mathbf{a}}\hat{\mathbf{b}} + \hat{\mathbf{b}}\hat{\mathbf{a}})\,. \tag{68}$$

So, for arbitrary nonzero vectors **a**, **b**

$$ab\cos\theta = \tfrac{1}{2}(\mathbf{ab}+\mathbf{ba})\,. \tag{69}$$

where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$.

7 Working with a Basis

Let $\{\sigma_i\}$ be a set of orthonormal basis vectors for the space of vectors. Then, by definition

$$\sigma_i\cdot\sigma_j = \delta_{ij} \quad i,j = 1,2,3\,, \tag{70}$$

where δ_{ij} is the Kronecker delta.

So, for vectors $\mathbf{a} = \sum_i a_i\sigma_i$ and $\mathbf{b} = \sum_j b_j\sigma_j$,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (\sum_i a_i\sigma_i)\cdot(\sum_j b_j\sigma_j) \\ &= \sum_i\sum_j a_ib_j\sigma_i\cdot\sigma_j \\ &= \sum_i\sum_j a_ib_j\delta_{ij} \\ &= \sum_i a_ib_i\,. \end{aligned} \tag{71}$$

[7]The proofs of the distributive property that I've seen in various sources either use geometric figures to examine projections or else use matrix representations of vectors, which implicitly employ a basis. I'm not against using a basis for some proofs. But I am against introducing a basis implicitly.

Thus, for $\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a}$, we get

$$\mathbf{a} \cdot \mathbf{a} = \sum_i a_i^2 = a_1^2 + a_2^2 + \ldots + a_n^2 \,, \tag{72}$$

a result well-known in Gibbs dot product.

8 Working with 'Spinors' of $\mathcal{G}^3$

In geometric algebra, we can represent the rotation of a vector $\mathbf{a}$ in 3D by a spinor R acting in this manner:[8]

$$\mathbf{a} \mapsto \mathbf{a}' = R^\dagger \mathbf{a} R \,, \tag{73}$$

where $R^\dagger R = 1$. We can also show that $RR^\dagger = 1$. Multiply both sides of $R^\dagger R = 1$ by the bilateral operator $R \,\_\, R^\dagger$ to get

$$R(R^\dagger R)R^\dagger = R^\dagger 1 R = 1 \,. \tag{74}$$

Or, on re-associating, we get $(RR^\dagger)^2 = 1$. Therefore,

$$RR^\dagger = \pm 1 \,. \tag{75}$$

It's a simple matter from here to show that $RR^\dagger = 1$.

Thus, it is a straightforward matter to prove that the dot product of two vectors is invariant under rotations in 3D by the following:

$$\begin{aligned} \mathbf{a}' \cdot \mathbf{b}' &= \tfrac{1}{2}(\mathbf{a}'\mathbf{b}' + \mathbf{b}'\mathbf{a}') \\ &= \tfrac{1}{2}(R^\dagger \mathbf{a} R R^\dagger \mathbf{b} R + R^\dagger \mathbf{b} R R^\dagger \mathbf{a} R) \\ &= \tfrac{1}{2}(R^\dagger \mathbf{ab} R + R^\dagger \mathbf{ba} R) \quad (\text{set} RR^\dagger = 1) \\ &= R^\dagger [\tfrac{1}{2}(\mathbf{ab} + \mathbf{ba})] R \\ &= \mathbf{a} \cdot \mathbf{b} R^\dagger R \\ &= \mathbf{a} \cdot \mathbf{b} \,. \end{aligned} \tag{76}$$

9 Trigonometric Relations

We can prove the Law of Cosines using either the dot product or the geometric product. Let's use the latter. Refer to Figure 1 for the setup. We start with the innocent-looking vector relation:

$$\mathbf{a} = \mathbf{b} + \mathbf{c} \,, \tag{77}$$

and then subtract $\mathbf{b}$ from both sides and square both sides:

$$(\mathbf{a} - \mathbf{b})^2 = \mathbf{c}^2 \,. \tag{78}$$

[8] See Chapter 5 of NFCM, [1], p. 280

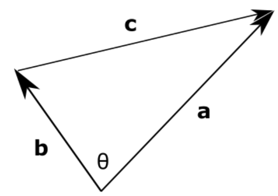

Figure 2. We can get the Law of Cosines from this triangle with the dot product or the geometric product.

Expanding the LHS of (78), and using (66), we get

$$\mathbf{a}^2 + \mathbf{b}^2 - (\mathbf{ab} + \mathbf{ba}) = \mathbf{a}^2 + \mathbf{b}^2 - 2\,\mathbf{a}\cdot\mathbf{b} = a^2 + b^2 - 2ab\cos\theta\,. \tag{79}$$

Equating this to the RHS of (78), we have that

$$a^2 + b^2 - 2ab\cos\theta = c^2\,. \tag{80}$$

From Cosine to Sine

Let $\mathbf{a}$ and $\mathbf{b}$ be any two unit vectors, i.e., vectors whose squares are unity. From (66), and that $\mathbf{a}\cdot\mathbf{b} = \cos\theta$, we get that

$$2\,\mathbf{a}\cdot\mathbf{b} = \mathbf{ab} + \mathbf{ba}\,, \tag{81a}$$

$$2\,\mathbf{a}\wedge\mathbf{b} = \mathbf{ab} - \mathbf{ba}\,. \tag{81b}$$

Squaring both sides (noting that $\mathbf{a}^2 = \mathbf{b}^2 = 1$), we get

$$4\cos^2\theta = \mathbf{abab} + \mathbf{abba} + \mathbf{baab} + \mathbf{baba} = \mathbf{abab} + 2 + \mathbf{baba}, \tag{82a}$$

$$4(\mathbf{a}\wedge\mathbf{b})^2 = \mathbf{abab} - \mathbf{abba} - \mathbf{baab} + \mathbf{baba} = \mathbf{abab} - 2 + \mathbf{baba}\,. \tag{82b}$$

Now, subtracting (82b) from (82a) and dividing through by 4, we get that

$$\cos^2\theta - (\mathbf{a}\wedge\mathbf{b})^2 = 1\,. \tag{83}$$

So, to force our geometric algebra to be consistent with trigonometry, we must demand another axiom:

Axiom of Trigonometric Consistency The theorems of geometric algebra produced in this paper shall be consistent with the theorems of Trigonometry.

We're now in a position to make a strong claim about the nature of $\mathbf{a}\wedge\mathbf{b}$:

$$(\mathbf{a}\wedge\mathbf{b})^2 \le 0 \quad \text{and} \quad |\mathbf{a}\wedge\mathbf{b}| = \sin\theta\,. \tag{84}$$

In fact, we can make an even stronger claim. For general vectors $\mathbf{c}$ and $\mathbf{d}$, we can claim

$$\mathbf{c}\wedge\mathbf{d} = |\mathbf{c}|\,|\mathbf{d}|\;\hat{\mathbf{c}}\wedge\hat{\mathbf{d}} \quad \text{and} \tag{85a}$$

$$|\mathbf{c}\wedge\mathbf{d}| = |\mathbf{c}|\,|\mathbf{d}|\sin\theta = cd\sin\theta\,. \tag{85b}$$

Putting these things together, we get

$$\mathbf{cd} = \mathbf{c}\cdot\mathbf{d} + \mathbf{c}\wedge\mathbf{d} = cd(\cos\theta + \sin\theta\,\imath)\,, \tag{86}$$

where $\mathbf{i}$ is a unit bivector in the plane spanned by $\mathbf{c}$ and $\mathbf{d}$.

10 Generalized Dot Product

Theorem:

For $r > 1$,

$$\mathbf{a}\cdot(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdot\wedge\mathbf{a}_r) = \sum_{k=1}^{r}(-1)^{k-1}\mathbf{a}\cdot\mathbf{a}_k\,\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdots\wedge\overset{\vee}{\mathbf{a}}_k\wedge\cdots\wedge\mathbf{a}_r\,, \tag{87}$$

where the $\vee$ means to delete this a_k from the line of wedges in the kth term.

Lemma 1:

Let A_k be a k-blade. Then

$$\langle\, A_k \,\rangle_1^\dagger = \langle\, A_k^\dagger \,\rangle_1. \tag{88}$$

Lemma 2:

Let $\mathbf{c}$ and $\mathbf{d}$ be vectors. Then

$$\mathbf{dc} = 2\mathbf{c}\cdot\mathbf{d} - \mathbf{cd}\,. \tag{89}$$

The proof of this is a simple algebraic manipulation of (9).

Proof of (87): (for case $r = 2$)

$$\mathbf{a}\cdot(\mathbf{a}_1\wedge\mathbf{a}_2) = \langle\,\mathbf{a}(\mathbf{a}_1\wedge\mathbf{a}_2)\,\rangle_1 \tag{90a}$$

$$= \langle\,\mathbf{a}\tfrac{1}{2}(\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}_2\mathbf{a}_1)\,\rangle_1 \tag{90b}$$

$$= \tfrac{1}{2}\langle\,\mathbf{a}\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}\mathbf{a}_2\mathbf{a}_1\,\rangle_1 \tag{90c}$$

$$= \tfrac{1}{2}\langle[2\mathbf{a}\cdot\mathbf{a}_1 - \mathbf{a}_1\mathbf{a}]\mathbf{a}_2 - [2\mathbf{a}\cdot\mathbf{a}_2 - \mathbf{a}_2\mathbf{a}]\mathbf{a}_1\rangle_1 \tag{90d}$$

$$= \mathbf{a}\cdot\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}\cdot\mathbf{a}_2\mathbf{a}_1 - \tfrac{1}{2}\langle\mathbf{a}_1\mathbf{a}\mathbf{a}_2 - \mathbf{a}_2\mathbf{a}\mathbf{a}_1\rangle_1 \tag{90e}$$

$$= \mathbf{a}\cdot\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}\cdot\mathbf{a}_2\mathbf{a}_1 - \tfrac{1}{2}[\,\langle\mathbf{a}_1\mathbf{a}\mathbf{a}_2\rangle_1 + \langle\mathbf{a}_2\mathbf{a}\mathbf{a}_1\rangle_1\,] \tag{90f}$$

$$= \mathbf{a}\cdot\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}\cdot\mathbf{a}_2\mathbf{a}_1\,, \tag{90g}$$

where

$$-\langle \mathbf{a}_1 \mathbf{a} \mathbf{a}_2 \rangle_1 + \langle \mathbf{a}_2 \mathbf{a} \mathbf{a}_1 \rangle_1 = 0\,, \tag{91}$$

because of Lemma 1.

11 Conclusion

> The vector space axioms do not mention norms, angles, or projections. Nor can these important geometric ideas be derived from the axioms. The *inner product* of two vectors (sometimes called the “dot” product) adds these concepts to the vector space formalism. ([3] p. 51.)

This above quote from Alan Macdonald, from his book *Linear and Geometric Algebra*, hits directly on the question of where the geometric content of geometric algebra comes from. This question is more at the heart of this paper than just to derive a few of the theorems concerning the dot product.

References

[1] D. Hestenes, *New Foundations in Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.

[2] D. Hestenes, *New Foundations for Mathematical Physics*, published by the author on-line, © 1998.

The address for the articles is at:

https://davidhestenes.net/geocalc/html/NFMP.html

[3] A. Macdonald, *Linear and Geometric Algebra*, (2010).