The Fibonacci Numbers in Brief

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1 Introduction

The Fibonacci numbers turn up in a variety of places, and are used to demonstrate a number of features of mathematics. But here we just get started on them.

2 Getting Started

Leonardo Fibonacci was a thirteenth-century mathematician who helped to convince the Europeans to convert to the decimal number system we now enjoy in mathematics. But his fame comes from his publishing results on the so-called Fibonacci numbers, which had been know in antiquity to the mathematicians of India.

The Fibonacci sequence is based on the recursive definition: Starting with 0 and 1, add these two numbers to get 1.[1] Add the 1 and the 1 to get 2. Add the 1 and the 2 to get 3. Add the 2 and the 3 to get 5, and we have the start of an infinite sequence of numbers: $\{0, 1, 1, 2, 3, 5, \ldots\}$. The sequence has the simple recursive formula

$$F_{n+2} = F_n + F_{n+1} \,, \tag{1}$$

where $F_0 = 0$, $F_1 = 1$, $F_2 = 0 + 1 = 1$, $F_3 = 1 + 1 = 2$, $F_4 = 1 + 2 = 3$, and so on. Given that this sequence is defined over the nonnegative integers, it has no ending and the numbers get big really fast:

$$\begin{aligned} &0, 1, 1,2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \\ &\quad 987, 1597, 2584, 4181, 6765, 10946, 17711, \\ &\qquad 28657, 46368, 75025, 121393, 196418, 317811, \ldots \end{aligned} \tag{2}$$

Let's first investigate the progressive consecutive ratios of these numbers, say, starting with 377:

$$610/377 = 1.618037, \quad 987/610 = 1.618033, \quad 1597/987 = 1.618034, \ldots \tag{3}$$

[1] The literature on Fibonacci numbers shows the starting number as either 0 or 1, but the recurrence definition for these numbers is not affected.

This new sequence of numbers looks like it may be approaching a famous number in mathematics, the **Golden Ratio**

$$\varphi = 1.61803398\ldots \quad \text{(not rounded)}. \tag{4}$$

Let's take a closer look at the Golden Ratio.[2]

According to the ancient Greeks, the Golden Ratio is said to occur as a relationship on the lengths of two adjacent sides of a rectangle[3] if, letting L stand for the length of the longer side and S the length of the shorter side, we get the proportion

$$\frac{L}{S} = \frac{L+S}{L}\,, \tag{5}$$

which is supposed to be a rectangle of maximal esthetic proportions (that is, of greatest beauty). Mind you, Equation (5) does not define the actual lengths of L and S, only their ratio.

Anyway, this equation has a numeric solution. Let $x = L/S$ and substitute back into (5) to get

$$x = 1 + x^{-1}\,. \tag{6}$$

Multiplying this through by x and rearranging terms, we get the usual form of a quadratic equation:

$$x^2 - x - 1 = 0\,. \tag{7}$$

The quadratic formula gives us the two irrational roots

$$\varphi_\pm = \frac{1 \pm \sqrt{5}}{2}\,, \tag{8a}$$

or in decimal form:

$$\varphi_+ = 1.618033988\ldots \quad \varphi_- = -0.618033988\ldots\,. \tag{8b}$$

[2] According to Wikipedia, this relationship of the ratios of consecutive Fibonacci numbers to the Golden Ratio was first discovered by Johannes Kepler.

[3] We can talk about the Golden Ratio without mention of rectangles. Apparently the ancient Greeks divided line segments into the Golden Mean and used that idea to design features of the Parthenon, but this is a controversial claim.