The GCD and LCM in Brief

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We must use time as a tool, not as a couch.
— John F. Kennedy

1 The GCD

The 'greatest common divisor' (GCD) of two positive integers a, b is the largest positive integer that both divides both a and b, evenly. The notation for this is $\mathrm{GCD}(a, b)$. When I need clarity on this concept myself, I remember that there is a set of all common divisors of a and b. So, two things to note about this set: First, it will always exists, even if it only has the element 1. Second, since this set is finite, it will always have a greatest element, namely the $\mathrm{GCD}(a, b)$.

An example of this is that the integer 2 divides both 4 and 8, evenly. Hence, 2 is in the set of common divisors of 4 and 8. But it's not the $\mathrm{GCD}(4, 8)$, because 4 is also in the set of common divisors of 4 and 8 and $4 > 2$. Also, it's easy to see that there is no larger integer that's a common divisor of 4 and 8, therefore,

$$\mathrm{GCD}(4, 8) = 4\,. \tag{1}$$

Now, for any choices for a and b positive integers, a common divisor of them both is, obviously, less than or equal to a and less than or equal to b. Let's look at a limiting case: Unity will always divide both a and b, hence, unity is always in the set of common divisors of the pair. What if unity is the largest common divisor? This is often a very important special case, i.e.,

$$\mathrm{GCD}(a, b) = 1\,. \tag{2}$$

Definition: When the $\mathrm{GCD}(a, b) = 1$, then we say that a and b are 'relatively prime' to each other, or that they are 'coprime'.

2 The LCM

The 'least common multiple' (LCM) of two positive integers a, b is the smallest positive integer that both a and b divide evenly. The notation for this is $\mathrm{LCM}(a, b)$.

Now, for any choices for a and b positive integers, a common multiple of them both is, obviously, their product, ab. But there are times that what we really want is the **smallest** number that they both divide evenly, or, the LCM of a and b.

Let's do an example. Consider $a = 8$ and $b = 12$. Their product is $ab = 8 \times 12 = 96$. Obviously, both 8 and 12 will divide 96 evenly. But can we do 'better'? Yes, the $\mathrm{LCM}(8, 12) = 24$.

3 The GCD-LCM Theorem

Let a and b be two positive integers. Then

$$\mathrm{GCD}(a, b) \cdot \mathrm{LCM}(a, b) = ab\,. \tag{3}$$

Corollary: If $\mathrm{GCD}(a, b) = 1$ then

$$\mathrm{LCM}(a, b) = ab\,. \tag{4}$$

As a practical matter, it may be easier in some problems to calculate the $\mathrm{GCD}(a, b)$ in order to calculate the $\mathrm{LCM}(a, b)$ and then apply the rule:

$$\mathrm{LCM}(a, b) = \frac{ab}{\mathrm{GCD}(a, b)}\,. \tag{5}$$

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4 The Final Comment

Of these two functions, the more commonly used one is the GCD, which has frequent usages in number theory and group theory.