Geometric Series

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January 29, 2025

Our greatest weakness lies in giving up. The most
certain way to succeed is always to try
just one more time.
—Thomas Edison

1 The Problem

Find the value of

$$\phi = a + ar^1 + ar^2 + ar^3 + ar^4 + ar^5 \,, \tag{1}$$

where a is a constant value. And if that's not hard enough, find the value of

$$\phi = a + ar^1 + ar^2 + ar^3 + ar^4 + ar^5 + \cdots + ar^n \,, \tag{2}$$

where n could be an arbitrarily large natural number. Perhaps even more intimidating is

$$\phi = a + ar^1 + ar^2 + ar^3 + ar^4 + ar^5 + \cdots \,. \tag{3}$$

Now, since the value of a finite number of terms, each of finite value, is finite, the first two summations are sure to be finite numbers, but not so for the third series. We'll deal with this issue in just a minute.

First, let's adopt the standard summation notation for these series, by adopting the $\sum$ symbol. Thus, the first three summations become, respectively:

$$\phi = \sum_{i=0}^{5} ar^i \,, \tag{4}$$

and

$$\phi = \sum_{i=0}^{n} ar^i \,, \tag{5}$$

and

$$\phi = \sum_{i=0}^{\infty} ar^i \,. \tag{6}$$

You could ask, "We've got in each summation the term ar^0, so what does that mean?" Well, we assume that r is never zero in all these series summations, so $r^0 = 1$, and therefore, $ar^0 = a$. One could also ask, "What kind of numbers are we dealing with here?" Generally, one works with either the real numbers or the complex numbers in these summation formulas.

Now, I want to make a simplification: For each of the summations above, I want to factor out the common factor of a, leaving us with

$$\phi = a \sum_{i=0}^{5} r^i \,, \tag{7}$$

and

$$\phi = a \sum_{i=0}^{n} r^i \,, \tag{8}$$

and

$$\phi = a \sum_{i=0}^{\infty} r^i \,. \tag{9}$$

Now, part of the definition of a geometric series is that fact that every pair of successive terms have the same ratio (called a 'common ratio'). Using the above series, that means that for the jth term of the series ($j > 0$), ar^j, its ratio with its preceding term, ar^{j-1} is

$$\frac{ar^j}{ar^{j-1}} = r \,, \tag{10}$$

which is the constant r.

2 Closed-Form Formulas

From this point on, I want to consider only those series in which $a = 1$. If you have a series in which it is not unity, just factor it out.

Now, a formula is said to be 'closed-form'[1] if it can be expressed as one or two terms (at least by appearances), usually at worst being a rational function.

I will now give the closed-form formulas (without proofs) for our two cases of interest, finite and infinite. For the finite-sum case:

$$1 + r + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r} \quad \text{where} \quad r \neq 1 \,, \tag{11}$$

and for the infinite-sum case:

$$1 + r + r^2 + \cdots = \sum_{i=0}^{\infty} r^i = \frac{1}{1 - r} \quad \text{where} \quad r \neq 1 \,, \tag{12}$$

and to guarantee that the infinite series converges, we require that $0 < |\, r \,| < 1$.

[1] This is my own working definition. Basically, a closed form expression takes up a lot less room on a line than what it's replacing. This is one of those terms in which short definitions are always wrong, but long definitions are always confusing.