

Introduction to Gibbs' Vector Calculus

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Abstract

Just the basics this time.

What I can't stand is people lookin' down on others
when they're not doin' them no harm.
— John Wayne in 'North to Alaska'

The Einstein Summation Rule: In an algebraic expression with indices, any time an index is repeated, it will be summed on, unless stated otherwise. For example

$$A_i B_i = \sum_{i=1}^3 A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3. \quad (1)$$

What a handy rule for making things more compact!

Typically, I will use $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ as a basis for Euclidean 3-space. Then, vector \mathbf{A} can be expanded thusly,

$$\mathbf{A} = \sum_{i=1}^3 A_i \boldsymbol{\sigma}_i = A_1 \boldsymbol{\sigma}_1 + A_2 \boldsymbol{\sigma}_2 + A_3 \boldsymbol{\sigma}_3. \quad (2)$$

An index that is summed on is referred to as **dummy**, meaning that it could be replaced by a different symbol, so long the replacement does not conflict with other indices being used in the expression. Thus, (1) could also be written as

$$A_k B_k = \sum_{k=1}^3 A_k B_k = A_1 B_1 + A_2 B_2 + A_3 B_3. \quad (3)$$

The vector derivative ∇ can be similarly expanded:

$$\nabla = \sum_{i=1}^3 \partial_i \boldsymbol{\sigma}_i = \partial_1 \boldsymbol{\sigma}_1 + \partial_2 \boldsymbol{\sigma}_2 + \partial_3 \boldsymbol{\sigma}_3, \quad (4)$$

where

$$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}, \quad (5)$$

with the independence rule:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}, \quad (6)$$

which is the Kronecker delta.

Also, we have that

$$\partial_i(AB) = (\partial_i A)B = A(\partial_i B) = A\partial_i B, \quad (7)$$

which is the Product Rule for differentiation.

Finally, we have that

$$\partial_i \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i \partial_i, \quad (8)$$

as the $\boldsymbol{\sigma}_i$'s are constant vectors.

Next, the cross product. The Gibbs' vector cross product is a creature of Euclidean 3-space. Just the same, we need to know how it relates to the wedge product. For vectors \mathbf{A}, \mathbf{B} ,

$$\mathbf{A} \wedge \mathbf{B} = i \mathbf{A} \times \mathbf{B}. \quad (9)$$

The symbol i in an expression is typically the unit pseudoscalar of Euclidean 3-space,

$$i = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3. \quad (10)$$

As an example, with \mathbf{k} independent of the x_i coordinates,

$$\partial_j \mathbf{x} \cdot \mathbf{k} = \partial_j (x_i k_i) = \delta_{ji} k_i = k_j. \quad (11)$$

If we wanted, we could replace the dummy index i by, say, ℓ or p , but not by j , because it's already taken.