Hyperbolic Trig Functions

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We must use time as a tool, not as a couch.
— John F. Kennedy

1 Hyperbolic Trigonometry Presentation

Think of this as a reference guide rather than a tutorial.

$$\cosh y + \sinh y = e^{y} \,, \tag{1a}$$

$$\cosh y - \sinh y = e^{-y} \,, \tag{1b}$$

$$\cosh^2 y - \sinh^2 y = 1 \,, \tag{1c}$$

$$\cosh \tfrac{1}{2} y = \sqrt{\frac{\cosh y + 1}{2}} \,, \tag{1d}$$

$$\sinh \tfrac{1}{2} y = \operatorname{sgn}(y) \sqrt{\frac{\cosh y - 1}{2}} \,, \tag{1e}$$

$$\frac{d}{dy} \cosh y = \sinh y \,, \tag{1f}$$

$$\frac{d}{dy} \sinh y = \cosh y \,. \tag{1g}$$

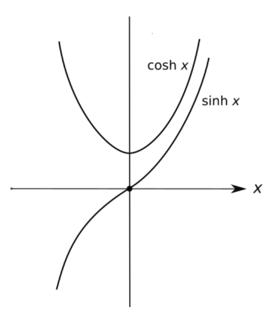

Figure 1. A sketch of the hyperbolic cosine and sine when x is real valued. Note that $\cosh x$ is an even function, $\sinh x$ is odd but has an inverse on the whole real line.

2 Basics of Complex Numbers with Trig and Hyperbolic Trig Functions

Let's begin with the Euler relations:

$$\cos\theta + i\sin\theta = e^{i\theta}\,, \tag{2a}$$

$$\cos\theta - i\sin\theta = e^{i\theta}\,, \tag{2b}$$

Next, let's invert them:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{i\theta})\,, \tag{3a}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{i\theta})\,, \tag{3b}$$

where, in the above cases, I used the usually understood real variable θ, but that can be replaced by the complex variable z. In fact, soon we will do so.

Okay, how to represent $\tan z$ by exponentials?

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\,. \tag{4}$$

Relation of the usual trig functions to the hyperbolic trig functions:

$$i\sin\theta = \sinh(i\theta)\,, \tag{5a}$$

$$\cos\theta = \cosh(i\theta)\,, \tag{5b}$$

$$i\tan\theta = \tanh(i\theta)\,, \tag{5c}$$

$$-i\cot\theta = \coth(i\theta)\,. \tag{5d}$$