Integration Techniques Paper 1

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Abstract

Herein is a collection of integration techniques for indefinite integrals. I found the problems from a variety of sources, most of which I no longer know where they came from.

1 Introduction

Indefinite integrals, when they exist, result in functions, whereas, definite integrals, when they exist, result in numbers. The purpose of this paper is to present some techniques to perform indefinite integrals. My major strategy will be to reduce or convert a given integral into a form which is in a robust integral table, though, I may attempt to reduce the integral as far as I can take it. The integrals are presented in no particular order.

Warning: This paper assumes that the reader has a basic knowledge of integration (such as integration by parts), along with a familiarity with trigonometric and hyperbolic identities, logarithms, exponentials, partial fractions – generally speaking, the stuff found in a course on Algebra 2.

Note: The symbol D_x means to differentiate by x.

2 Reference Integrals and Derivatives for Later Use

First, the derivatives:

$$D_x \sin x = \cos x \,. \tag{1}$$

$$D_x \cos x = -\sin x \,. \tag{2}$$

$$D_x \sec x = \sec x \tan x \,. \tag{3}$$

$$D_x \tan x = \sec^2 x \,. \tag{4}$$

Now, the integrals:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \,. \tag{5}$$

$$\int \ln x\, dx = \ln x\,(x-1) + C \,. \tag{6}$$

$$\int \frac{du}{u} = \ln u + C \,. \tag{7}$$

$$\int \sec x\, dx = \ln \Big| \sec x + \tan x \Big| + C \,. \tag{8}$$

3 Virtual Emplacement

So much of mathematics employs tricks that get used over and over in a wide variety of subject areas, yet go unnamed, and thus are hard to explain to one's readers when one uses them. Decades ago I invented the term *virtual emplacement* to refer to the algebraic action of adding a zero to an expression, or multiplying or dividing an expression by unity, or more generally, performing some function and its inverse to an expression, such as

$$x\sqrt{x+y} = \sqrt{x^2(x+y)} \quad \text{when} \quad x, y \geq 0\,. \tag{9}$$

Let's look at the expression $\dfrac{x^2}{1+x^2}$. Can we simplify it?[1] Yes, by performing a virtual emplacement.

$$\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2} = \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}\,. \tag{10}$$

Stuff like this comes up all the time in integration.

Now, as a real application, consider the integral:

$$\int \frac{x^2}{1+x^2}\,dx = \,? \tag{11}$$

Clearly, we can use the table integral (5), if we can message it into the correct form. But we can do that by employing the result of (10), as well, to get

$$\begin{aligned} \int \frac{x^2}{1+x^2}\,dx &= \int 1\,dx - \int \frac{dx}{1+x^2}\,, \\ \text{hence} \quad \int \frac{x^2}{1+x^2}\,dx &= x - \tan^{-1} x + C\,, \end{aligned} \tag{12}$$

where C is an arbitrary constant.

4 The 'Carrington' (Differential) Equation vs. Integration by Parts

Integration by Parts is one of the most used techniques in the bag of gimmicks of indefinite integration. The technique provides us with an integration identity:

$$\int u\,dv = vu - \int v\,du\,. \tag{13}$$

The proof of it is based on the product rule of differentiation:

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x)\,. \tag{14}$$

Now, we integrate:

$$f(x)g(x) = \int g(x)D_x f(x)\,dx + \int f(x)D_x g(x)\,dx\,, \tag{15}$$

or,

$$f(x)g(x) = \int g(x)\,df + \int f(x)\,dg\,, \tag{16}$$

[1] By 'simplify' in this case, I mean to reduce the given expression to a sum of expressions, each of which is easier to integrate than the original expression.

Now, let $u = f(x)$ and $v = g(x)$, then this last equation becomes

$$uv = \int v\, du + \int u\, dv\,, \tag{17}$$

from which follows (13).

The Carrington[2] differential equation, associated to some integral, is any equation of the form[3]

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x) \tag{18}$$

that facilitates an integration problem. The next step is to integrate across (18).

This Carrington equation is not unique, though I will sometimes proffer to the reader 'the Carrington equation', which should be interpreted merely as the *particular* Carrington equation that I chose.

Heuristic: I often set $g(x)$ to be the integrand of the integral and then set $f(x) = x$.

Let's now do an example problem: Find the integral[4]

$$I = \int x \ln x\, dx\,, \tag{19}$$

We start with the Carrington differential equation (which follows the above heuristic):

$$D_x[x^2 \ln x] = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x\,. \tag{20}$$

Now we integrate:

$$\begin{aligned} x^2 \ln x &= 2I + \int x\, dx \\ &= 2I + \frac{1}{2}x^2\,. \end{aligned} \tag{21}$$

From this we get that

$$2I = x^2 \ln x - \frac{1}{2}x^2 + C'\,, \tag{22}$$

or

$$\int x \ln x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\,. \tag{23}$$

5 Some Example Integrals

Problem 1:

Find the integral

$$I = \int \sqrt{x^2 - x + 1}\, dx\,. \tag{24}$$

My solution will be to transform this integral into the form

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2 + a^2}\right| + C\,. \tag{25}$$

[2]For the moment, this name is a place holder until I can come up with a better name for it. Dr. Carrington refers to a fictional character in the scifi movie *The Thing from Another World*, 1951. This equation is merely an ansatz and is, in most cases, not unique.

[3]The LHS of the Carrington Equation could contain a product of three or more factors, but that gets messy fast.

[4]I will often use the variables I, J, K, etc as placeholders for integrals to reduce the visual mess of the problem.

To accomplish this, I need to complete the square of the radicand $x^2 - x + 1$ by

$$x^2 - x + 1 = (x + A)^2 + B^2 = x^2 + 2Ax + A^2 + B^2\,, \tag{26}$$

so that $A = -1/2$ and $B^2 = 3/4$. Therefore,

$$\begin{aligned} I = \int \sqrt{x^2 - x + 1}\, dx &= \int \sqrt{(x - 1/2)^2 + 3/4}\, d(x - 1/2) \\ &= \frac{x - 1/2}{2}\sqrt{(x - 1/2)^2 + 3/4} + \frac{3}{8}\log\left|(x - 1/2) + \sqrt{(x - 1/2)^2 + 3/4}\right| + C\,. \end{aligned} \tag{27}$$

Problem 2:

Find the integral

$$I = \int \ln\sqrt{1 + x^2}\, dx\,. \tag{28}$$

My first transformation is to this

$$I = \tfrac{1}{2}\int \ln(1 + x^2)\, dx\,. \tag{29}$$

I begin with the Carrington equation

$$D_x[x\ln(1 + x^2)] = \ln(1 + x^2) + \frac{2x^2}{1 + x^2}\,. \tag{30}$$

Next, we divide through by 1/2 and then integrate:

$$\tfrac{1}{2}x\ln(1 + x^2) = I + \int \frac{x^2}{1 + x^2}\, dx\,. \tag{31}$$

Now, we solve this for I and employ (12), to get

$$I = \tfrac{1}{2}x\ln(1 + x^2) - x + \tan^{-1} x + C\,. \tag{32}$$

And we're done.