Lambert W Functions

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Recently, the W(x) function amassed quite a following in the
mathematical community. Its most faithful proponents
are suggesting elevating it among the present set
of elementary functions, such as sin(x),
cos(x), exp(x), ln(x), etc.
— Darko Veberic
('Lambert W Function for Applications
in Physics' [preprint])

1 Introduction to the Lambert W Function

We'll start off with an overview of the Lambert W function. It was invented to unravel this product in variable x:

$$xe^x = A\,, \tag{1}$$

to solve for x. What a cool function, indeed! Hence,

$$x = W(xe^x) = W(A)\,, \tag{2}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with. And that's it, at least for us here.

The Lambert W function is to the expression xe^x what the natural logarithm is to e^x: They are both designed to extract from each the value/s of x.

I intend to present a brief introduction to the Lambert W function. Wikipedia (among other sources) do a better job. My point of view on them is that of an olympiad-style math problem-solver, which is rather specific — at least at this time. So, let's get started.

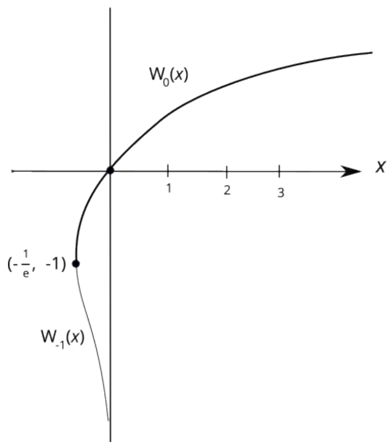

Figure 1. A rough sketch of the Lambert W function when x is real-valued. Emphasis is made of its multi-valued behavior near the origin. On the interval $-1/e < x < 0$, $W(x)$ has two distinct real values, depicted as $W_0(x)$ and $W_{-1}(x)$, which can be determined (approximately) with a computer program, such as WolframAlpha (using ProductLog[]).

From Wikipedia: Some special cases for the Lambert W function:

$W_0(e) = W(1 \cdot e^1) = 1.$

$W_0(-e^{-1}) = -1.$

$W_0(e^{1+e}) = e.$

$W_0(\frac{e^{1/2}}{2}) = 1/2.$

$W_0(\frac{e^{1/n}}{n}) = 1/n.$

$W_0(1) \equiv \Omega = e^{-W_0(1)} = -\ln W_0(1) \approx 0.567143.$

$W_0(-1) \approx -0.31813 + 1.33723i.$

$W_0(-\pi/2) = i\pi/2.$

$W_0(x^{x+1} \ln x) = x \ln x.$

$W_0(0) = W(0 \cdot e^0) = 0.$

$$W_0(z \ln z) = \ln z \quad \text{where} \quad \left(x \geq \frac{1}{e} \approx 0.36788\right). \tag{3}$$

2 Extrication

Extrication is the name I'm giving to the process of replacing an expression that contains W with an expression that does not. There are two ways to do this:

In some very special cases, one can extract out of $W(\cdot)$ if the dot is one of a few numbers like those we saw above: 0, 1, e, etc, in which case the expression is replaced by a number of some sort.

The other way can occur under very special cases of an expression in x, say, such as xe^x, $x^{x+1}\ln x$, or $x\ln x$, which has the extrications:

$$W_0(xe^x) = x\,, \tag{4}$$

$$W_0(x\ln x) = \ln x \quad \text{where} \quad (x \geq \frac{1}{e} \approx 0.36788)\,, \tag{5}$$

$$W_0(x^{x+1}\ln x) = x\ln x\,. \tag{6}$$

3 Lemma 1: Changing Base

The basic problem I have when solving a problem that is 'almost in the correct form' to apply an extrication above, but not quite. Much time can be consumed performing a boring variable transformation that places the problem in the correct 'form'. Let me give an example. The following is problem 317 of Math Diversions that I solved in the Pset:

Given the relation

$$a + 3125^a = 0\,, \tag{7}$$

find the values of a. Now, if it were of the form $a + e^a = 0$, it would be a bit easier to solve, right? The following lemma let's us deal efficiently with (7).

Lemma 1:

Let s be a real number such that $s > 0$. Then, given

$$zs^z = B\,, \tag{8}$$

then

$$z = W_s(B)\,, \tag{9}$$

where

$$W_s(B) \equiv \frac{W(B\ln s)}{\ln s}\,, \tag{10}$$

which becomes the ordinary Lambert W function when $s = e$, which I refer to as the 'Lambert W function base s'.[1]

Proof:

Let

$$s^z = e^y\,, \tag{11}$$

and then take the logarithm:

$$z\ln s = y\,, \tag{12}$$

[1] This notation I invented myself.

and then solve for z,

$$z = \frac{y}{\ln s}\,. \tag{13}$$

Next, we substitute this stuff back into (8), to get

$$ye^y = B\ln s\,. \tag{14}$$

Now, when we take the Lambert W function across this equation, we have that

$$y = W(B\ln s)\,. \tag{15}$$

On returning to the variable z, we get

$$z = \frac{W(B\ln s)}{\ln s}\,. \tag{16}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(B\ln e)}{\ln e} = W(B)\,, \tag{17}$$

which is the usual Lambert W function.

Warning: Please don't get confused when you see me replace the base e with the base s in a calculation, as above. If you see W_0, W_{-1}, or W_n, these are the standard notations for various forms of the Lambert W function. In any case, my use of $W_s(B)$ is so non-standard, that I would (probably) only use it within an internal calculation, not as an end result.

Now, let's solve the problem above. We can begin by rewriting (7) as

$$-a = 3125^a\,, \tag{18}$$

or

$$-a\,3125^{-a} = 1\,. \tag{19}$$

We can now solve for $-a$:

$$-a = W_{3125}(1) = \frac{W(1\cdot\ln 3125)}{\ln 3125} = \frac{W(\ln 3125)}{\ln 3125}\,. \tag{20}$$

But $3125 = 5^5$, so

$$a = -\frac{W(5\ln 5)}{5\ln 5}\,. \tag{21}$$

If we want to be very general, we can write for the answer

$$a = -\frac{W_n(5\ln 5)}{5\ln 5} \quad \text{for} \quad n \in \mathbb{Z}\,, \tag{22}$$

which I shall not go into here. But for the principal value, which is (21), we can use another lemma (that we'll state and prove next) to simplify:

$$a = -\frac{W(5\ln 5)}{5\ln 5} = -\frac{\ln 5}{5\ln 5} = -\frac{1}{5}\,. \tag{23}$$

Clearly, using this lemma has saved us quite a few perfunctory steps.

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4 Lemma 2: Extricating $x \ln x$ from $W(\cdot)$

I'm referring to the principal value of W, given $W(x\ln x)$ as in (5). Anyway, if

$$x\ln x = B\,, \tag{24}$$

then

$$\ln x = W(B)\,. \tag{25}$$

Proof: Let $x = e^y$, then

$$W(x\ln x) \to W(e^y(y)) = W(ye^y) = y = \ln x\,. \tag{26}$$

Let's do an example problem.

Given the relation

$$x^x = e^{-\pi + i\ln 4}\,, \tag{27}$$

find the values of x.

We can begin by taking the natural logarithm across (27):

$$x\ln x = -\pi + i\ln 4 + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{28}$$

At first thought, it may seem as though applying the Lambert W function across this equation is taking a step backwards. I mean, what do we do with $W(x\ln x)$? But that's the beauty of it. Out of $W(x\ln x)$ we can extricate just $\ln x$.

Then we can apply the Lambert W function across this equation and use the lemma above, to get

$$\ln x = W(-\pi + i\ln 4 + 2\pi i n) \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{29}$$

Lastly, we just need to raise e to the power of Eq. (29),[2] to get

$$x = e^{W(-\pi + i\ln 4 + 2\pi i n)} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{30}$$

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[2] Yes, it sounds strange, but it works for me. The expression 'to take an object to the power of an equation' can be explained this way: Let our equation be 'LHS = RHS', and let our object be z. Then "z raised to the power 'LHS = RHS'" means the following:

$$z^{\text{LHS}} = z^{\text{RHS}}\,.$$

This is often particularly useful when dealing with logarithms.

5 Lemma 3: Extricating $x^{x+1} \ln x$ from $W(\cdot)$

The identity

$$W(x^{x+1} \ln x) = x \ln x \tag{31}$$

was in the list I pulled from Wikipedia. I list it here for completeness, though I have not used it so far.

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6 Lemma 4: $W(A)e^{W(A)} = A$

Let's begin with the relation

$$xe^x = A\,. \tag{32}$$

Then

$$x = W(xe^x) = W(A)\,. \tag{33}$$

If we now make the change of variable:

$$x = \ln y\,, \tag{34}$$

then (33) becomes

$$\ln y = W(y \ln y) = W(A)\,. \tag{35}$$

In other words, besides trying to conform the given expression to that of (32), we can alternatively conform it to this:

$$y \ln y = A\,, \tag{36}$$

Now, multiply (35) through by y:

$$y \ln y = yW(A)\,. \tag{37}$$

Applying the transitive property to these last two equations, yields

$$A = yW(A)\,. \tag{38}$$

But $y = e^x$, so then

$$A = e^x\, W(A) = e^{W(A)}\, W(A)\,, \tag{39}$$

where we used (33). Anyway, we have that

$$W(A)\, e^{W(A)} = A\,. \tag{40}$$

If my memory serves me correctly, I have used this identity to convert an answer that WolframAlpha got to compare it to the answer I got.

7 Supplemental Problem#1

Given the relation

$$3^k = k^9 \,, \tag{41}$$

find the solutions of k.

8 Solution

Part #1:

Let $k = 3^\alpha$, then we have that

$$3^{3^\alpha} = 3^{9\alpha} \,. \tag{42}$$

On setting the exponents equal, we get

$$3^\alpha = 9\alpha = 3^2\alpha \,. \tag{43}$$

After a little algebra, we have that

$$3^{\alpha-2} = \alpha \,. \tag{44}$$

This equation can be solved by inspection to get

$$\alpha = 3 \,. \tag{45}$$

Therefore

$$k = 3^3 = 27 \,. \tag{46}$$

Part #2:

To look for solutions other than just integers, we proceed as follows:

Start by taking the natural logarithm across (56), to get

$$k \ln 3 = 9 \ln k \,, \tag{47}$$

which, after a little algebra, becomes

$$\frac{1}{9} \ln 3 = k^{-1} \ln k \,. \tag{48}$$

Now we multiply through by -1:

$$-\frac{1}{9} \ln 3 = -k^{-1} \ln k = k^{-1} \ln k^{-1} \,. \tag{49}$$

Next, we take the Lambert W function across this equation to get

$$W(-\frac{1}{9} \ln 3) = \ln k^{-1} \,. \tag{50}$$

Solving this for k^{-1}, we have that

$$k^{-1} = e^{W(-\frac{1}{9}\ln 3)} \,. \tag{51}$$

If we wanted, we could solve for k by taking the multiplicative inverse, but I want the answer in the form that WolframAlpha poses it. So, I multiply through by $W(-\frac{1}{9}\ln 3)$, to get

$$W(-\frac{1}{9}\ln 3)\, k^{-1} = W(-\frac{1}{9}\ln 3)\, e^{W(-\frac{1}{9}\ln 3)} \,. \tag{52}$$

Next, we use the identity:

$$W(z)\, e^{W(z)} = z \,. \tag{53}$$

Then, (52) becomes

$$W(-\frac{1}{9}\ln 3)\, k^{-1} = -\frac{1}{9}\ln 3 \,. \tag{54}$$

After some algebra and generalizing the solution, we have that

$$k = -9\frac{W_n(-\frac{1}{9}\ln 3)}{\ln 3} \quad \text{where } n \in \mathbb{Z} \,. \tag{55}$$

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9 Lambert Problem and the ProductLog[] of WolframAlpha

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Source: https://www.youtube.com/watch?v=q2XyddOgPjo&t=2s
Title: The Seemingly Impossible Equation That Has a
        Beautiful Solution
 Presenter: Mental Math
```

Given the relation

$$\pi^x = x^\pi \tag{56}$$

solve for real values of x.

By inspection, we can see that one solution is $x = \pi$. Just the same, there are two real solutions and we'll find them both by the same method, which I'd say looks like a job for the Lambert W function!

10 Solution

We'll begin by taking the $1/\pi$ root on both sides of (56), to get

$$(\pi^x)^{1/\pi} = x \tag{57}$$

or

$$(\pi^{1/\pi})^x = x \,. \tag{58}$$

For simplification, let us define

$$\beta \equiv \pi^{1/\pi} = 1.4396\,, \tag{59}$$

where I used a calculator, so then

$$\beta^x = x\,. \tag{60}$$

Next, we employ a little algebra to get

$$1 = x\beta^{-x}\,. \tag{61}$$

Then we multiply through by -1 to put the equation in the appropriate form.

$$-1 = -x\beta^{-x}\,. \tag{62}$$

Now we flip sides and the apply Lambert W function base β: [3]

$$-x = W_\beta(-1)\,. \tag{63}$$

Next, we multiply through by -1 and use one of the lemma in the Appendix, we get

$$x = -\frac{W_n(-1\cdot\ln\beta)}{\ln\beta} = -\frac{W_n(-0.364)}{0.364}\,, \tag{64}$$

where n is an integer. The two real values occur for $n = -1$ and for $n = 0$.

I used WolframAlpha to calculate this Lambert W_n function for me, using the separate commands

`ProductLog[-1,-0.364]` and also `ProductLog[0,-0.364]`.

$$\text{ProductLog}[n, -0.364] = \begin{cases} -1.15276 & \text{for } n = -1\,, \\ -0.8614 & \text{for } n = 0\,. \end{cases} \tag{65}$$

On substituting these values into (64), we have that

$$x = \begin{cases} 3.1676 & \text{for } n = -1\,, \\ 2.366 & \text{for } n = 0\,. \end{cases} \tag{66}$$

WolframAlpha calculated the values to get

$$x = \begin{cases} 3.14159 & \text{for } n = -1\,, \\ 2.38218 & \text{for } n = 0\,. \end{cases} \tag{67}$$

So we found the approximate value for π, which is the trivial solution, and we also get the value 2.38218, which is the non-trivial solution.

[3] For explanations about what I'm doing, see the Appendix or see my short write up in PDF on the Lambert W function.