Math Diversions, Problem 1

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> The human mind has never invented a labor-saving
> machine equal to algebra.
> — J. Willard Gibbs

1 Problem

This problem is found on the YouTube channel **PreMath**, from December 25, 2021. My solution here is a little different from that given by the presenter.

Statement of the problem:

Given the relation

$$x + \frac{1}{x} = 5\,, \tag{1}$$

find the value of

$$x^5 + \frac{1}{x^5} = ? \tag{2}$$

Comment: Eq. (1) is a quadratic equation in x, and as such we can solve for x by use of the quadratic formula. But here we will solve for the value of the expression in (2) without first solving for x.

Solution to the problem:

Note that

$$\left(x + \frac{1}{x}\right)^2 = 25\,, \tag{3}$$

which reduces to

$$x^2 + \frac{1}{x^2} = 23\,. \tag{4}$$

Using Eq. (1), we have that

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) = 5 \cdot 23 = 115\,. \tag{5}$$

The LHS of this last equation expands to

$$x^3 + \frac{1}{x^3} + (x + \frac{1}{x}) = x^3 + \frac{1}{x^3} + 5\,. \tag{6}$$

Thus (5) reduces to

$$x^3 + \frac{1}{x^3} = 110\,. \tag{7}$$

On multiplying (4) by (7), we get that

$$\Big(x^2 + \frac{1}{x^2}\Big)\Big(x^3 + \frac{1}{x^3}\Big) = 23 \cdot 110 = 2530\,. \tag{8}$$

And this partially reduces to

$$\Big(x^5 + \frac{1}{x^5}\Big) + \Big(x + \frac{1}{x}\Big) = 2530\,. \tag{9}$$

One more simplification gives us the result we're looking for:

$$x^5 + \frac{1}{x^5} = 2525\,. \tag{10}$$

Done.

Comment: It might be interesting to work the problem backwards, by which I mean to start with (10) and then find the value of $x + 1/x$.