Math Diversion Problem 10

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> We must use time as a tool, not as a couch.
> — John F. Kennedy

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=yyBOvZTRb7I
Title:An Interesting Exponential Expression
 Presenter: SyberMath
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1 The Problem

Given the relation

$$\phi = e^{2^{\frac{\ln(\ln 2)}{\ln 2}}}\,, \tag{1}$$

simplify ϕ over the real numbers.

2 Preparation

Just in case the reader could use a refresher on logarithms,

$$\log_a b = \frac{\log_c b}{\log_c a}\,. \tag{2}$$

where a, b, c are positive real numbers. And

$$\log_a a = 1\,. \tag{3}$$

3 The Solution

Lemma:

$$\ln 2 = \log_e 2 = \frac{\log_2 2}{\log_2 e} = \frac{1}{\log_2 e}\,. \tag{4}$$

And also

$$\frac{\ln(\ln 2)}{\ln 2} = \frac{\log_2(\ln 2)}{\log_2 2} = \log_2(\ln 2)\,. \tag{5}$$

We'll begin by taking the natural logarithm to both sides:

$$\ln \phi = 2^{\frac{\ln(\ln 2)}{\ln 2}} , \tag{6}$$

and then the logarithm base 2:

$$\log_2 \ln \phi = \frac{\ln(\ln 2)}{\ln 2} . \tag{7}$$

Next, we use the result from (5), to get

$$\log_2 \ln \phi = \log_2 (\ln 2) , \tag{8}$$

On canceling the logarithms base 2, we get

$$\ln \phi = \ln 2 , \tag{9}$$

from which we get

$$\phi = 2 . \tag{10}$$