

Math Diversion Problem 104

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Success is failing 19 times and succeeding the 20th.

— Julia Andrews

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=Z1DFHJqUziE>

Title: Solving a nice integral - Solution to past exam

Presenter: Kmath addict

1 The Problem

Find the value of the following definite integral:

$$\int_0^1 \ln(1+x^2) dx. \quad (1)$$

2 The Solution

My plan is to use the standard identity

$$\int \ln w \, dw = w(\ln w - 1) + c, \quad (2)$$

in this solution.

Notation:

$$\bar{I} \equiv \int_0^1 \ln(1+x^2) dx, \quad (3)$$

but

$$I \equiv \int \ln(1+x^2) dx, \quad (4)$$

with the intent that I will work mostly with the indefinite integral I (ignoring the arbitrary constant) until the final steps.

$$I = \int \ln(1+x^2) dx = \int \ln(1+ix)(1-ix) dx \quad (5a)$$

$$= \int \ln(1+ix) dx + \int \ln(1-ix) dx \quad (5b)$$

$$= -i \int \ln(1+ix) d(ix) + i \int \ln(1-ix) d(-ix) \quad (5c)$$

$$= -i \int \ln(1+ix) d(1+ix) + i \int \ln(1-ix) d(1-ix) \quad (5d)$$

$$= -i \int \ln(w_+) d(w_+) + i \int \ln(w_-) d(w_-), \quad (5e)$$

where $w_+ \equiv 1+ix$ and $w_- \equiv 1-ix$.

Then,

$$I = -iw_+(\ln w_+ - 1) + iw_-(\ln w_- - 1) \quad (6a)$$

$$= -i(1+ix)(\ln(1+ix) - 1) + i(1-ix)(\ln(1-ix) - 1) \quad (6b)$$

$$= i[(\ln(1-ix) - \ln(1+ix))] + x[(\ln(1+ix) + \ln(1-ix))] - i(-i+x) - (i+x) \quad (6c)$$

$$= i \ln \frac{(1-ix)}{(1+ix)} + x \ln(1+x^2) - 2x \quad (6d)$$

$$= i \ln \frac{(1-ix)^2}{(1+x^2)} + x \ln(1+x^2) - 2x. \quad (6e)$$

Now,

$$\bar{I} = I_0^1 = I(1) - I(0) = I(1). \quad (7)$$

So,

$$\bar{I} = i \ln \frac{(1-i)^2}{2} + \ln 2 - 2 \quad (8a)$$

$$= i \ln \left(\frac{(1-i)}{\sqrt{2}} \right)^2 + \ln 2 - 2 \quad (8b)$$

$$= i \ln \left(e^{-i\pi/4} \right)^2 + \ln 2 - 2 \quad (8c)$$

$$= i \ln e^{-i\pi/2} + \ln 2 - 2 \quad (8d)$$

$$= i(-i\pi/2) + \ln 2 - 2 \quad (8e)$$

$$= \pi/2 + \ln 2 - 2. \quad (8f)$$

This is the same answer that WolframAlpha presented.