Math Diversion Problem 109

P. Reany

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> Truth, like oil, will in time rise to surface.
> — Charlie Chan

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=EO1d9GxEcaI
Title: Harvard MIT Math Tournament | HMMT 2018 |
 Presenter: Math Gold Metalist
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1 The Problem

Given the relation

$$x^{2x^2} = 3\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{2}$$

then the Lambert W function base a is defined as

$$z = W_a(B)\,, \tag{3}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{4}$$

which becomes the ordinary Lambert W function when $a = e$.

3 The Solution

I want to solve this by making my standard substitution of

$$x^2 = 3^\alpha\,, \tag{5}$$

then (1) becomes

$$(3^{\alpha})^{3^{\alpha}} = 3^{1}\,. \tag{6}$$

Setting exponents equal, we get

$$\alpha 3^{\alpha} = 1\,. \tag{7}$$

On taking the Lambert W function base 3, we have that

$$\alpha = W_3(1) = \frac{W(1 \cdot \ln 3)}{\ln 3} = \frac{W(\ln 3)}{\ln 3}\,. \tag{8}$$

Thus,

$$x^2 = 3^{\alpha} = e^{\ln 3^{\alpha}} = e^{\alpha \ln 3} = e^{\frac{W(\ln 3)}{\ln 3}\,\ln 3} = e^{W(\ln 3)}\,, \tag{9}$$

Hence, the real solutions are

$$x = \pm e^{\frac{1}{2}W(\ln 3)}\,. \tag{10}$$