Math Diversion Problem 118

P. Reany

January 24, 2025

> It is my experience that proofs involving matrices can be
> shortened by 50% if one throws the matrices out.
> — Emil Artin

> More than a century since its debut, representation theory
> has served as a key ingredient in many of the most important
> discoveries in mathematics. Yet its usefulness
> is still hard to perceive at first.
> — Kevin Hartnett

(Representation theory — among other uses, it is the representation of elements of an arbitrary group by the elements of a linear map to a vector space. Once a basis is chosen, the linear map can take the form of a matrix group.)

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=wuMld8B9d5k
Title: A Nice Math Olympiad Problem
 Presenter: Maths Black Board
```

1 The Problem

Given the relations

$$\sqrt{a} + \sqrt{2-b} = \sqrt{2}\,, \tag{1a}$$

$$\sqrt{b} + \sqrt{2-a} = \sqrt{2}\,, \tag{1b}$$

find the values of a, b.

2 The Solution

We begin by re-arranging both sides of both given relations, and then squaring:

$$a = 2 - 2\sqrt{2}\sqrt{2-b} + (2-b)\,, \tag{2a}$$

$$b = 2 - 2\sqrt{2}\sqrt{2-a} + (2-a)\,. \tag{2b}$$

Next, if we subtract the second of these from the first, we get (eventually)

$$\sqrt{2-b} = \sqrt{2-a}\,, \tag{3}$$

implying that

$$a = b\,. \tag{4}$$

Thus, (2a) can be rewritten as

$$a = 2 - 2\sqrt{2}\sqrt{2-a} + (2-a)\,, \tag{5}$$

which becomes

$$a = 2 - \sqrt{2}\sqrt{2-a}\,, \tag{6}$$

which becomes with a bit of algebra

$$a(a-2) = 0\,. \tag{7}$$

Thus we have the solution pairs:

$$(a,b) = (0,0)\,,\ (2,2)\,. \tag{8}$$