Math Diversion Problem 122

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'Obvious' is the most dangerous word in mathematics.
— Eric Temple Bell

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Ycyag8jY1TQ
Title: So, you want a HARD math question with exponents?
 Presenter: Higher Mathematics
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1 The Problem

Given the relation

$$x^2 = 4^x \,, \tag{1}$$

find the real values of x.

2 The Solution

Right off the bat, this should look to the reader like a Lambert W function solution. Let's see if it is. One form for the Lambert W function is this:

$$W(ze^z) = z \quad \text{where} \quad z \geq 0 \,. \tag{2}$$

We begin by taking the square root of both sides of (1):

$$x = \pm 2^x \,. \tag{3}$$

So, we need to replace 2^x by e^y, or better yet, by e^{-y}, and $y = -x \ln 2$. Hence,

$$\frac{-y}{\ln 2} = \pm e^{-y} \,, \tag{4}$$

which can be rewritten as

$$ye^y = \pm \ln 2 \,. \tag{5}$$

Applying the Lambert W function across this equation, we get

$$y = W(ye^y) = W(\ln 2) \,. \tag{6}$$

Therefore,

$$x = -\frac{W(\ln 2)}{\ln 2}\,. \tag{7}$$