

Math Diversion Problem 135

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

The definition of a good mathematical problem is the mathematics
it generates rather than the problem itself.

— Andrew Wiles

The YouTube video is found at:

Source: https://www.youtube.com/watch?v=67234_o6z-4

Title: Germany | A nice Logarithmic Math Olympiad Problem

Presenter: Super Academy

1 The Problem

Given the relation

$$x^{\log 27} + 9^{\log x} = 36, \quad (1)$$

find the values of x over the real numbers. By the way, I interpret the logarithm as being in base 10.

2 The Solution

The first thing I did was to introduce a change of variable. Let

$$x = 10^\alpha, \quad (2)$$

so that on substituting, we get

$$(10^\alpha)^{\log 27} + 9^{\log 10^\alpha} = 36, \quad (3)$$

and this simplifies to

$$27^\alpha + 9^\alpha = 36. \quad (4)$$

Let's introduce another variable change:

$$y = 3^\alpha, \tag{5}$$

then, (4) becomes

$$y^3 + y^2 - 36 = 0. \tag{6}$$

When I tried some small integers for y , I found that $y = 3$ is one solution.

Then, on factoring (6), we get

$$(y - 3)(y^2 + 4y + 12) = 0. \tag{7}$$

The roots to the quadratic are complex, so I will ignore them.

Thus, for $y = 3$, we get,

$$\alpha = \frac{\log y}{\log 3} = \frac{\log 3}{\log 3} = 1. \tag{8}$$

Hence,

$$x = 10^1 = 10. \tag{9}$$