Math Diversion Problem 136

P. Reany

January 24, 2025

Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

> You may have to fight a battle more than once to win it.
> — Margaret Thatcher

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=MRTDBSwlH2I
Title: Harvard University Admission Exam Tricks
 Presenter: Super Academy
```

1 The Problem

Given the relation

$$64^x = x^{192}\,, \tag{1}$$

find the values of x over the real numbers.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{4}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{5}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{6}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

3 The Solution

The first thing I did was to take the 192th root across the equation, to get

$$\alpha^x = x\,, \tag{7}$$

where

$$\alpha = 64^{1/192}\,, \tag{8}$$

where α is to be the Lambert base. After a little algebra, we have that

$$-x\alpha^{-x} = -1\,. \tag{9}$$

Thus,

$$-x = W_\alpha(-1) = \frac{W(-1\cdot\ln\alpha)}{\ln\alpha}\,. \tag{10}$$

But

$$\ln\alpha = \ln\left(64^{1/192}\right) = \frac{1}{192}\ln 2^6 = \frac{\ln 2}{32}\,, \tag{11}$$

thus (10) becomes:

$$x = -\frac{32W(-\frac{1}{32}\ln\alpha)}{\ln 2}\,. \tag{12}$$

[1] This notation I invented myself.