

Math Diversion Problem 143

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The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=PvE7HiRakQk>

Title: A Nice Algebra Problem | Math Olympiad

Presenter: SALogic

1 The Problem

Given the relation

$$x^x = 4^{4+x}, \quad (1)$$

find the values of x over the real numbers.

2 The Solution

So, I'll make my usual variable transformation in this situation.

$$x = 4^\alpha. \quad (2)$$

Then

$$(4^\alpha)^{4^\alpha} = 4^{4+4^\alpha}, \quad (3)$$

which can be rewritten as

$$4^{\alpha 4^\alpha} = 4^{4+4^\alpha}. \quad (4)$$

On equating exponents, we have (with some manipulation) that

$$(\alpha - 1)4^{\alpha-1} = 1. \quad (5)$$

My first impulse was to try an integer, in this case 1, but that didn't work. My second impulse was to try a rational number, which turns out to be $\alpha = \frac{3}{2}$, therefore

$$x = 4^{\frac{3}{2}} = 8, \quad (6)$$

which does work. The fastest way to prove this is to rewrite (1) in base 2 rather than in base 4.