

Math Diversion Problem 156

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You can be creative in anything — in math, science, engineering,
philosophy — as much as you can in music or in
or in painting or in dance.

— Ken Robinson

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=U5Svh3S7Ziw>

Title: A Nice Math Olympiad Algebra Problem

Presenter: Master T Maths Class

1 The Problem

Given the relation

$$x^{\sqrt{x}} = 10, \tag{1}$$

find the values of x over the real numbers.

2 The Solution

I like to approach this kind of problem in a standard way. So, let

$$x = 10^\alpha. \tag{2}$$

Then (1) becomes

$$(10^\alpha)^{(10^\alpha)^{1/2}} = 10, \tag{3}$$

which simplifies to

$$10^{\alpha 10^{\alpha/2}} = 10^1. \tag{4}$$

On equating exponents, we have that

$$\alpha 10^{\alpha/2} = 1. \tag{5}$$

So, how does one approach this equation to solve for α ? My method is first to look for an integer solution and then a rational solution. If that fails, one

can always move things in the direction of Lambert's W function,¹ which I'll do now. Let's divide through by 2:²

$$\frac{1}{2}\alpha 10^{\alpha/2} = \frac{1}{2}. \quad (6)$$

Let $\beta = \alpha/2$, then we get

$$\beta 10^\beta = \frac{1}{2}. \quad (7)$$

Okay, we're getting closer. Next, introduce y so that it satisfies the relation

$$e^y = 10^\beta, \quad (8)$$

from which we get that

$$\beta = \frac{y}{\ln 10}. \quad (9)$$

Next, we substitute these relations into (7):

$$\frac{y}{\ln 10} e^y = \frac{1}{2}. \quad (10)$$

Therefore,

$$y = W(ye^y) = W\left(\frac{\ln 10}{2}\right). \quad (11)$$

Going back to (9), we have that

$$\beta = \frac{W(\frac{1}{2} \ln 10)}{\ln 10}. \quad (12)$$

So,

$$\alpha = 2 \frac{W(\frac{1}{2} \ln 10)}{\ln 10} = \frac{W(\frac{1}{2} \ln 10)}{\frac{1}{2} \ln 10}. \quad (13)$$

Therefore,

$$x = 10^{\frac{W(\frac{1}{2} \ln 10)}{\frac{1}{2} \ln 10}}. \quad (14)$$

Or, you can do something like this:

$$x = 10^\alpha = 10^{2\beta}, \quad (15)$$

so,

$$\sqrt{x} = 10^\beta = e^y = e^{W(\frac{1}{2} \ln 10)}. \quad (16)$$

Hence

$$x = e^{2W(\frac{1}{2} \ln 10)}. \quad (17)$$

¹The Lambert W function has the remarkable property that (under certain domain restrictions) $W(ze^z) = z$.

²By the way, make variable substitutions as often as you think necessary.