Math Diversion Problem 171

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> The only reason that we like complex numbers is
> that we don't like real numbers.
> — Bernd Sturmfels

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=uguB0e_8ygs
Title: A Natural Log Problem With Complex Numbers
 Presenter: aplusbi
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1 The Problem

Given the relation

$$\ln\left(iz + \sqrt{1 - z^2}\right) = \frac{i\pi}{3}\,, \tag{1}$$

find the values of z over the real numbers.

2 The Solution

My idea is to use the following identity in the hyperbolic functions:

$$\ln\left(x + \sqrt{1 + x^2}\right) = \sinh^{-1} x\,, \tag{2}$$

where x can be complex. So, first I have to transform into this form. Let $w = iz$. Then (1) becomes

$$\ln\left(w + \sqrt{1 + w^2}\right) = \sinh^{-1} w = \frac{i\pi}{3}\,. \tag{3}$$

Thus

$$iz = w = \sinh\frac{i\pi}{3}\,, \tag{4}$$

$$z = \frac{1}{i}\sinh\frac{i\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\,, \tag{5}$$

and, yes, I used another hyperbolic identity.