Math Diversion Problem 185

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Algebraic geometry seems to have acquired the reputation
of being esoteric, exclusive, and very abstract, with
adherents who are secretly plotting to take over
all the rest of mathematics. In one respect
this last point is accurate.
— David Mumford
[Not if the Category theorists have
have anything to do about it.]

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=psS7fG_kFd8
Title: An Imaginary Exponential Equation
    Problem 346
 Presenter: aplusbi
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1 The Problem

Given the relation

$$i^{z+i} = 1\,, \tag{1}$$

find the complex values of z. (Skip down to the solution, if you like.)

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{2}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (2) is

$$\overline{z} = a - bi\,. \tag{3}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2\,. \tag{4}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a\,. \tag{5}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2}e^{i\theta} = (z\overline{z})^{1/2}e^{i\theta} = r\,e^{i\theta}\,, \tag{6}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |z|^2\,. \tag{7}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

Lemma 1: If a complex number z is equal to its own conjugate $z = \overline{z}$, it's real.

Lemma 2: If a complex number z is complex conjugated twice then there's no change: $\overline{\overline{z}} = z$.

Lemma 3: The complex conjugated of a product or a sum is the product or sum of the complex conjugates: $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$.

Lemma 4: If $s, t \in \mathbb{R}$ and $z = s + ti$ then

$$i\overline{z} = t + si\,. \tag{8}$$

3 Basics of Complex Numbers with Trig Functions

Let's begin with the Euler relations:

$$\cos\theta + i\sin\theta = e^{i\theta}\,, \tag{9a}$$

$$\cos\theta - i\sin\theta = e^{i\theta}\,, \tag{9b}$$

Next, let's invert them:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{i\theta})\,, \tag{10a}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{i\theta})\,, \tag{10b}$$

where, in the above cases, I used the usually understood real variable θ, but that can be replaced by the complex variable z. In fact, soon we will do so.

Okay, how to represent $\tan z$ by exponentials?

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\,. \tag{11}$$

4 The Solution

We begin by replacing in (1) the i as base with $e^{i\pi/2}$ and unity by $e^{2ni\pi}$ for n an integer, then:

$$(e^{i\pi/2})^{z+i} = e^{2ni\pi}\,, \tag{12}$$

or

$$e^{2\pi i[\frac{1}{4}(z+i)]} = e^{2ni\pi}\,. \tag{13}$$

On setting the exponents equal, we get

$$\tfrac{1}{4}(z+i) = n\,. \tag{14}$$

Solving for z, we have that

$$z = 4n \ - i\,. \tag{15}$$