

Math Diversions, Problem 2

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Mental toughness is essential to success.

— Vince Lombardi

1 Problem

This problem is found on the YouTube channel **PreMath**, from December 23, 2021. My solution here is a little different from that given by the presenter.

Statement of the problem:

Given the relations

$$\frac{x^3 + y^3}{x + y} = 7, \quad (1)$$

and

$$\frac{x^3 - y^3}{x - y} = 19, \quad (2)$$

Find the possible real (integer) values of x and y .

Solution to the problem:

My experience in solving for the roots to the general cubic equation led me try to solve this problem by multiplying the two equations together, as

$$\frac{x^3 + y^3}{x + y} \frac{x^3 - y^3}{x - y} = 7 \times 19, \quad (3)$$

which reduces to

$$\frac{x^6 - y^6}{x^2 - y^2} = 133. \quad (4)$$

Now for a simplifying variable substitution:

$$\alpha = x^2, \quad \beta = y^2, \quad (5)$$

to get

$$\frac{\alpha^3 - \beta^3}{\alpha - \beta} = 133. \quad (6)$$

But we have a formula for the difference of two cubes, namely

$$\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2). \quad (7)$$

On applying this identity to the equation before it, we get that

$$\alpha^2 + \alpha\beta + \beta^2 = 133. \quad (8)$$

Let's put this equation in standard quadratic form and assume that α is our unknown to solve for. Then

$$\alpha^2 + \beta\alpha + (\beta^2 - 133) = 0. \quad (9)$$

Then, by use of the quadratic formula:

$$\alpha = \frac{-\beta \pm \sqrt{\beta^2 - 4(\beta^2 - 133)}}{2}, \quad (10)$$

which can be rewritten as

$$\alpha = \frac{-\beta \pm \sqrt{532 - 3\beta^2}}{2}. \quad (11)$$

Now, I'm going to make a simplifying assumption that both α and β are positive integers, and this implies that the discriminant of the squareroot of the last equation is a perfect square, which we should be able to find by exhaustive search on values of β . I say this with confidence because if the discriminant gets too large then the squareroot goes imaginary, which we have assumed must not happen.

Because of how α and β were defined in terms of x and y , we need only consider perfect-square values of α and β :

β	$532 - 3\beta^2$	$\sqrt{532 - 3\beta^2}$	$\alpha = (-\beta + \sqrt{532 - 3\beta^2})/2$
1	529	23	11
4	484	22	9 ✓
9	289	17	4 ✓
16	-236	—	—

So, we have our two perfect squares to work with. We can choose $\alpha = 4$ and $\beta = 9$, which gives us for values of x and y

$$x = \pm 2 \quad \text{and} \quad y = \pm 3, \quad (12)$$

which gives us the four pairs of solutions:

$$(2, 3), (2, -3), (-2, 3), (-2, -3), \quad (13)$$

but because of the symmetry of how α and β entered into (8), we can also switch the order of these ordered pairs to get

$$(3, 2), (-3, 2), (3, -2), (-3, -2). \quad (14)$$

At this point, I entered the two original equations and input all eight of these ordered pairs into WolframAlpha.com, but found that only four of them worked. These four were:

$$(2, 3), (-2, -3), (3, 2), (-3, -2). \quad (15)$$

Done.